ELEMENTARY ELECTRICAL CALCULATIONS

A MANUAL OF SIMPLE ENGINEERING MATHEMATICS, COVERING THE WHOLE FIELD OF DIRECT CURRENT CALCULATIONS, THE BASIS OF ALTERNATING CURRENT MATHEMATICS, NETWORKS AND TYPICAL CASES OF CIRCUITS, WITH APPENDICES ON SPECIAL SUBJECTS

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PREFACE

This book is designed to give in simple form what may be termed a foundation for the study of electrical calculations. The operations described require only an elementary knowledge of mathematics. It is a feature of electrical science that although it is built up on a basis of mathematics, a great part of the engineering calculations is comprised within the limits of arithmetic, while elementary algebra carries it a long way further. The algebra required in every day electrical work is so simple that it may be learned in a very short time, and it is perfectly fair to say that many use it daily without realizing that they do so. Some there are who might even be repelled from the subject if told that algebra was required in its operations, yet such may employ Ohm's law without realizing that it is expressed as an algebraic equation.

Very few calculations in this book call for the use of higher algebra than is involved in the treatment of Ohm's law, and the arithmetic employed is simpler than that used in commercial calculations. Where more advanced algebra is required, as in the solution of net-works, the matter is placed towards the end. The complex variable and the graphic solutions of alternating current problems are not employed, as they are outside the scope of the book.

In the Appendix some illustrations of geometric, of more involved algebraic, and of simple calculus solutions and demonstrations are given, which may interest those with some knowledge of the higher mathematics.

The author desires to express his acknowledgment and appreciation of the painstaking work of Mr. Newman D. Waffle, which has done much for the accuracy of the text and treatment of the subject.

The kindness of readers who will inform the author of any errors or misprints to be corrected in future editions will be greatly appreciated.

T. O'CONOR SLOANE.

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ELEMENTARY ELECTRICAL CALCULATIONS

HOW MADE AND APPLIED

CHAPTER I.

INTRODUCTORY.

Arithmetic. — Logarithms. — The Slide Rule. — Trigonometry. — Indication of Division. — Reciprocals. — Symbols. — Subscripts. — Meaning of *n*. — Algebraic Symbols. — Algebra. — Positive and Negative. — Algebraic Addition. — Algebraic Subtraction. — Coefficients. — Functions. — Calculus. — Electric Measurements. — Quantity. — Current. — Electro-motive Force. — Resistance. — The Watt. — Equivalence of Units. — Numerical Values.

Arithmetic. In practical electrical calculations much can be done by arithmetic. The operations may be abridged by various special methods applying the properties of numbers. The following are examples.

To multiply by 5, add a cipher and divide by 2. To multiply by 25, add two ciphers and divide by 4. To square any even number between 12 and 24, square one-half the number and multiply by 4. To square any number which is divisible by 3, not exceeding 36, square one-third of the number and multiply by 9. To multiply numbers which can be factored into small factors, the factors can be multiplied together. Thus $63 \times 49 = 7 \times 9 \times 7 \times 7 = 3087$.

There are other methods similar to these which can be utilized for mental calculations.

Special attention should be paid to exponential notation, as it plays an important part in dimensional formulas and deductions and in notation by powers of 10.

Cancellation can sometimes be used to great advantage;

often it is practically unavailable. Inspection of the problem will decide whether it is applicable or not.

Where decimals enter into a problem it is a question to be determined in each case how far they should be carried out. In the present book they are carried out far enough for most purposes of electrical work, which in this respect is not as a rule very exacting.

Logarithms. Familiarity with logarithms should be acquired. In many cases they facilitate operations; in the extraction of roots they are often indispensable.

The Slide Rule. The slide rule is a favorite instrument of the electrical and mechanical engineer. It gives practical results with great rapidity, but its limitations should be kept in mind. If of pocket size, it gives only approximate results, not as close as those given by logarithms, whose principles are carried out by it.

Trigonometry. Trigonometric functions are used to a limited extent in alternating-current work and in deductions of laws. Tables of natural sines and tangents are accurate enough for most purposes. Some familiarity with analytical trigonometry is desirable, and at least a knowledge of the ordinary trigonometric functions should be possessed by the student.

Indication of Division. There are various ways of indicating the division of one quantity by another. Suppose the division of 4 by 3 is to be indicated, it may be done as follows: $4 \div 3$, $\frac{4}{3}$, and 4×3^{-1}. Each of these expressions may be correctly read "four divided by three." This is the best way to express a fraction, as it keeps before the mind something often inadequately realized, namely, that a fraction is an indication of division. The fraction $\frac{4}{3}$ may be correctly called four-thirds; it should never be called "four over three."

When the constituents of a fraction are symbols the only

proper designation is the one employing the words "divided by." $\frac{a}{b}$ is to be expressed as "*a* divided by *b*," not "*a* over *b*."

Reciprocals. The reciprocal of an integral number is a fraction having 1 for its numerator and the number for its denominator. The reciprocal of a fraction is the fraction inverted. The reciprocal of 3 is $\frac{1}{3}$; the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. If the quantity is a mixed number reduce to a fraction and invert it.

Example. What is the reciprocal of (1) $\frac{a}{b}$, of (2) $4\frac{2}{3}$, and of (3) $(x + y)$?

Solution. (1) $\frac{b}{a}$, (2) $\frac{3}{14}$ $(4\frac{2}{3} = \frac{14}{3})$, and (3) $\frac{1}{(x + y)}$.

Symbols. To avoid the writing out of names, symbols are employed, such as e.m.f. for electro-motive force. To a considerable extent the same symbols are used by all electricians. In formulas single-letter symbols are used, such as E for electro-motive force.

Subscripts. When the same letter has to be used for different places the places may be designated by letters or numbers. Then the letter of the symbol is used with the distinguishing character placed at its foot as a subscript.

Suppose a current of electricity goes through a number of conductors designated as 1, 2, 3, and so on, and that resistance is denoted by the capital letter R. Then the resistance of each conductor would be denoted respectively by R_1, R_2, R_3, and so on.

Meaning of *n*. Sometimes a series of quantities, the number of which quantities is undetermined, has to be indicated. This is done by the use of the letter *n*, thus: Suppose resistances were to be indicated as above; it would be done thus: R_1, R_2, $R_3 \ldots R_n$. If *n* were once determined or settled on, a number could be substituted for it. Thus R_1, R_2, $R_3 \ldots R_6$ means a series of such "*R*'s," up to R_6. The first or "*n* series" means the same up to R_n.

Algebraic Symbols. Algebraic symbols are used in the deduction and expression of relations between quantities whose numerical values are not expressed or are unknown. Ohm's law in one form reads: Current is equal to electro-motive force divided by resistance. The algebraic expression of the law is

$$I = \frac{E}{R}.$$

If for any two of these symbols numerical values are substituted, the simple arithmetical operation indicated will give the value of the third.

Algebra. Algebra is the shorthand of arithmetic, and calculus is the shorthand of algebra. In both branches of mathematics the power and scope of the lower ones are also extended. While a great part of the necessary operations of electrical calculation can be done by arithmetic, some use of algebra is to be recommended. Sufficient knowledge of algebra for most purposes can be acquired in a very few weeks.

Positive and Negative. Quantities sometimes have positive values. Such are written without any sign. In formulas involving addition the sign is expressed. If a quantity has a negative value the sign must be prefixed, as -4, $-E$. Suppose that two e.m.f.'s were opposed to each other so that one tended to produce a current in one direction and the other e.m.f. tended to produce a current in the other direction. Then we could express this condition of things by giving a positive value to one e.m.f. and a negative sign to the other, as E and $-E$.

Algebraic Addition. Algebraic addition is the adding together of quantities, taking their signs into account. To do it, when quantities have different signs, add together all quantities of the same sign, thus getting two sums. Subtract the numerically smaller one from the larger one and give the remainder the sign of the larger of the two original sums. If

all the numbers to be added are of the same sign, add all together and prefix the sign of the original number to the sum if negative.

Example. Add together 5, 7, -4, -2 and 2.

Solution. Proceeding as above we have

$$\begin{array}{rlrlr}
5 & \text{and} & -4 & \text{and subtracting} & 14 \\
7 & & -2 & & 6 \\
2 & & \overline{} & & \overline{} \\
\overline{} & & -6 & & 8 \\
14 & & & &
\end{array}$$

to which a positive value is assigned, because the sign of the larger of the two sums, 14, is positive.

Algebraic Subtraction. Algebraic subtraction is done by changing the sign of the number to be subtracted and then adding the two algebraically.

Example. Subtract -9 from -3.

Solution. -9 is the subtrahend, or number to be subtracted. Changing its sign we have to add 9 and -3 algebraically to carry out the operation described. The algebraic sum is 6, whose sign is positive because the sign of the larger of the two quantities is positive.

In algebraic subtraction it is immaterial whether the number to be subtracted is larger or smaller than the number from which it is to be subtracted.

Coefficients. A coefficient is the multiplier of a symbol. Suppose the symbol r indicated 4 ohms; then in the expression $7r$, 7 would be the coefficient of r, and the expression would indicate 7×4 ohms $= 28$ ohms.

Functions. If a quantity multiplied or divided by a number gives a product or quotient equal to another quantity, the latter quantity is a function of the first. This is expressed algebraically in this way: $y = f(x)$, $a = f(b)$, reading "y is a function of x" and "a is a function of b." Suppose that $a = 3b$, then a would be a function of b. The same would be

the case if $a = b/3$. If $y = 2x^2$, then y would be a function of x^2, or $y = f(x^2)$.

Calculus. Many electrical deductions are easily reached by calculus. Some of these are given in a separate chapter.

Electric Measurements. — Quantity. Electricity may be measured by any one of its various effects. A quantity of electricity which will precipitate 0.001,118 gramme of silver is the practical or working measure of electric quantity and is called the coulomb.

Current. Under proper conditions what is called a current of electricity passes through a conductor. A current of such intensity that one coulomb per second passes is called a current of one ampere strength or intensity.

A 110-volt 16 candle power lamp uses about one-half an ampere when burning.

E.M.F. The conditions for producing a current include a conductor, such as a wire, and the maintenance of a potential difference between the ends of the conductor. The difference of potential is called potential difference, potential drop, and electro-motive force. The latter is abbreviated into e.m.f. Sometimes the capitals are employed, thus, E.M.F. The measure of e.m.f. is the volt for all ordinary purposes of engineering.

Resistance. A volt will produce a greater or less current through a conductor according to the size and material of the conductor. A volt will produce a current of almost one ampere through a copper wire No. 10 gauge and 1,000 feet long.

A conductor through which an e.m.f. of one volt will produce a current of one ampere is said to have a resistance of one ohm.

The Watt. If the number of volts acting on a conductor are multiplied by the amperes of current produced, the result expresses the number of a compound measure present, which

measure is called the volt-ampere or watt. The watt is a unit of rate of electric energy.

A watt will raise a pound nearly $\frac{3}{4}$ foot per second. A watt acting for a second is a volt-coulomb, which is the unit of electric energy and is equal to a joule. The two names are nearly synonymous, and in many cases can be used interchangeably.

Equivalence of Units. All units of force and energy in whatever systems can be reduced one to the other by the use of multipliers or divisors, which latter are called equivalents. Thus from the table of equivalents we can find the value of the watt in heat or in mechanical units and the value of the dyne in other units of force. But for the electrical units of current and of electro-motive force no equivalents exist. The same, with some slight reservation, may be said of other electric units.

Numerical Values. There are a number of numerical values which it is well to memorize. The following are examples. Each branch has its own set of constants and values, and it is a question of individual memory how many can be learned and retained.

30 feet a second, a single rail length, is about 20 miles an hour. 60 feet a second, a double or trolley rail length, is about 40 miles an hour. An ordinary railroad passenger car may be taken as 60 feet long for this computation.

1.47 feet in a second is 1 mile an hour.

The acceleration of gravitation is about 32.2 feet, 981 cm.

$\sqrt{2} = 1.414$; $\frac{1}{\sqrt{2}} = 0.707$.

Copper wire 1 foot long, 1 mil. diameter, has 10.79 ohms resistance.

Copper wire No. 10 A.W.G., 1,000 feet long, has about 1 ohm resistance.

746 watts = 1 horse power.

$\pi = 3.14159$; for approximate calculations, $3\frac{1}{7}$, $\frac{22}{7}$.

$2\pi = 6.2832$; $4\pi = 12.5664$; $\frac{\pi}{4} = 0.7854$; 1 radian $= 57.296°$.

CHAPTER II.

EXPONENTIAL NOTATION.

Powers. — Exponents. — Roots. — Symbols of Roots. — Fractional Exponents. — Decimal Exponents. — Negative Exponents. — Multiplication of Powers. — Division of Powers. — Extraction of Roots by Logarithms. — Powers of Ten. — Changing Exponents. — Addition and Subtraction of Powers. — Factoring Exponents.

Powers. A power of a number is a number produced by multiplying the original number by itself any number of times.

Exponents. An exponent is the index of a power. It indicates multiplication by itself of the quantity to which it is affixed, such multiplication to be repeated as many times as there are units in the exponent less one. Thus 2^5 indicates that 2 shall be multiplied by 2 four times. Writing it out it becomes $2 \times 2 \times 2 \times 2 \times 2 = 32$. It is seen that an exponent indicates the number of times a number is used as a factor in producing another number. In 32 the number 2 appears 5 times as a factor, hence its exponent is 5.

Roots. A root of a number is a number which multiplied by itself a certain number of times, as above, will produce the original quantity. 2 is a root of 4, because if multiplied by itself it will produce 4, thus, $2 \times 2 = 4$. A root of a higher order than the third root (cube root) is designated numerically; an index number is assigned all roots, which number indicates how often the root enters as a factor into the multiplication producing the original quantity. Thus 2 is the third root of 8, because if taken as a factor three times it gives 8, thus, $2 \times 2 \times 2 = 8$.

The second root of a quantity is usually called the square root, and the third root is usually called the cube root. Other roots are specified by number, as the fourth root, fifth root, and so on.

Symbols of Roots. A root is indicated by the radical sign ($\sqrt{\ }$) with the index number above the opening when higher than the second place. This is the usual method in arithmetic. Thus $\sqrt{4}$ indicates the square root or the second root of 4, namely 2; $\sqrt[4]{81}$ indicates the fourth root of 81, namely 3, for $3 \times 3 \times 3 \times 3 = 81$. But in electrical work the system of indicating roots by fractional exponents is generally employed. A root is indicated by a fractional exponent whose numerator is unity and whose denominator is the index of the root. The two roots given above would be indicated thus: $4^{\frac{1}{2}} = 2$ and $81^{\frac{1}{4}} = 3$. As a matter of nomenclature they may be called four to the one-half power and eighty-one to the one-fourth power.

Fractional Exponents. Exponents may be fractional, with any number in the numerator. The signification of a fractional exponent with a numerator of 1 has been explained. A fractional exponent in general indicates the root of a power of the number to which it is affixed. The power is the one indicated by the numerator of the fraction, the root is the one indicated by the denominator of the fraction.

Example. Calculate the value of $4^{\frac{2}{4}}$.

Solution. It is the fourth root of the second power of 4, or the second power of the fourth root of 4. It is immaterial which order is followed. Sometimes one is easier than the other. Performing the operations, we have $4^2 = 16$. This is the second power of 4, because $4 \times 4 = 16$. The fourth root of 16 is 2, because $2 \times 2 \times 2 \times 2 = 16$. Therefore $4^{\frac{2}{4}} = 2$. $4^{\frac{2}{4}}$ may be written in other ways. Its exponent $\frac{2}{4}$ can be reduced to its lowest terms and can be written $\frac{1}{2}$ or 0.5. Substituting these expressions for $\frac{2}{4}$ gives $4^{\frac{2}{4}} = 4^{\frac{1}{2}} = 4^{0.5} = 2$, as before.

Decimal Exponents. An exponent consisting of a decimal fraction can be expressed as a vulgar fraction. Thus $10^{0.7001} =$

$10^{\frac{7001}{10000}}$. From a table of logarithms the above is found to be 5.012.

Negative Exponents. A negative exponent indicates the reciprocal of the indicated power of the number.

Example. Express the value of $5^{-3} = \frac{1}{5^3}$.

Solution. $5^3 = 125$. The reciprocal, $\frac{1}{5^3}$, is $\frac{1}{125}$ which is the value of 5^{-3}.

Multiplication of Powers. The product of identical numbers with exponents affixed is equal to the number raised to the power indicated by the algebraic sum of the exponents.

Example. Multiply 6^3 by 6^2.

Solution. The sum of the exponents is 5. Then $6^3 \times 6^2 = 6^5 = 7776$.

Example. Multiply 6^3 by 6^{-2}.

Solution. The sum of the exponents is 1, giving as result $6^1 = 6$.

Example. Multiply 6^{-3} by 6^2.

Solution. The sum of the exponents is -1, and $6^{-1} = \frac{1}{6}$.

This can be done in another way. $6^{-3} = \frac{1}{216}$. $6^2 = 36$. $\frac{1}{216} \times 36 = \frac{36}{216} = \frac{1}{6}$, as before.

Example. Apply the same process to multiplying 6^3 by 6^{-2}.

Solution. $6^3 = 216$. $6^{-2} = \frac{1}{36}$. $\frac{216}{36} = 6$, as in the example above.

Example. Multiply $3^{\frac{3}{2}}$ by $3^{\frac{1}{2}}$.

Solution. The sum of the exponents is $\frac{4}{2} = 2$. The product is then $3^2 = 9$.

Division of Powers. The quotient of identical numbers with exponents affixed is equal to the same number with a new

exponent, which is the algebraic difference of the two exponents, that of the dividend being taken first as minuend.

Example. Divide 4^4 by 4^2.

Solution. Subtracting the exponents for the new exponent and affixing it to the original number gives

$$4^4 \div 4^2 = 4^2.$$

Doing the same by arithmetical process, we have $4^4 = 256$ and $4^2 = 16$. $256 \div 16 = 16 = 4^2$, as before.

Example. Divide 4^3 by 4^4.

Solution. Proceeding as above we find for the new exponent $3 - 4 = -1$. The quotient then is the original number, 4, with an exponent -1, or 4^{-1}, $= \frac{1}{4}$.

Applying ordinary arithmetic, we have $4^3 = 64$, $4^4 = 256$, and $64 \div 256 = \frac{64}{256} = \frac{1}{4}$, as before.

Example. Divide 5^7 by 5^6.

Solution. The quotient is 5. No exponent is expressed because it is unity and positive, and such exponent is not written out but is understood.

Extraction of Roots by Logarithms. To extract a root use logarithms, except for the square root, whose extraction is so simple that it is applicable for computation. In using logarithms for this as for other purposes pay strict attention to the characteristic. Do not guess at the decimal place. If the logarithm to be divided by the index of the root has a negative characteristic proceed as follows. Multiply 10 by the divisor, in the present case by the index of the root, and place it after the mantissa of the logarithm with a negative sign between. Subtract numerically the characteristic of the logarithm from the same product and place it in front of the logarithm as a new characteristic in place of the original one, giving it a positive sign.

Example. Divide the logarithm $\bar{2}.25042$ by 7.

Solution. $10 \times 7 = 70$, which is to follow the mantissa with a minus sign between. $70 - 2 = 68$, which is the new characteristic of the logarithm, and we have the division $(68.25042 - 70) \div 7 = 9.75006 - 10 = \bar{1}.75006$.

Powers of Ten. Notation by powers of ten is simply a special case of exponential notation, subject to the rules as given. The exponent of a power of ten indicates the number of ciphers there would be in its development. Thus 10^3 indicates 1000, 10^6 indicates 1,000,000, and so on.

Example. Write 34,000,000,000 in powers of ten.

Solution. It is 34 multiplied by 10 raised to the power indicated by the number of ciphers. There are 9 ciphers; therefore it is written 34×10^9.

Example. Multiply 34×10^6 by 27×10^9.

Solution. $34 \times 27 \times 10^{15} = 918 \times 10^{15}$.

For all operations of multiplication and division in powers of ten, follow the rules of ordinary arithmetic as far as the numbers other than the powers of ten are concerned. For the powers of ten follow the rules given for exponential notation.

Example. Divide 34×10^9 by 27×10^6.

Solution. Divide 34 by 27 and multiply the quotient by 10 raised to the power indicated by the exponent obtained by subtracting 6 from 9, which is 3. This gives $\frac{34}{27} \times 10^3$.

Example. Divide 29×10^7 by 21×10^9.

Solution. $29 \div 21 = 1.38$. The new exponent is $7 - 9 = -2$, giving as the quotient 1.38×10^{-2}, which, if we depart from powers of ten, is 0.0138.

Changing Exponents. The exponent of the ten factor may be changed by changing the location of the decimal point of the number. If the exponent is diminished, the decimal point of the number must be moved to the right as many points as the units removed from the exponent. If the exponent is

increased, the decimal point of the number is moved to the left a corresponding number of points. This process enables us to remove the decimal point from the number if it is introduced by any operation.

Example. Remove the decimal point from 1.38×10^4.

Solution. If the exponent of 10^4 is reduced by 2, the decimal point of the number must be moved two points to the right. This will remove the decimal point. Thus $1.38 \times 10^4 = 138 \times 10^2$.

Example. Reduce 1.996×10^{-7} to an expression without a decimal.

Solution. We see that the exponent of 10 must be diminished by 3. This is done by algebraic subtraction. $-7-3 = -10$, and $1.996 \times 10^{-7} = 1{,}996 \times 10^{-10}$.

Addition and Subtraction of Powers. Exponential notation applies principally to the operations of multiplication and division. Numbers of different exponents cannot be added or subtracted. If such operation is to be done, the numbers must be reduced to factors of the same exponents, and the numbers with exponents above unity must be identical. In many cases this is impossible, at which times the operation can only be indicated.

Example. Add 3×10^9 to 56×10^6.

Solution. $3 \times 10^9 = 3{,}000 \times 10^6$. This can obviously be added to 56×10^6; the sum is $3{,}056 \times 10^6$.

Example. Subtract 56×10^6 from 3×10^9.

Solution. Reducing as above, we obtain $(3{,}000 \times 10^6) - (56 \times 10^6) = 2{,}944 \times 10^6$ as the remainder.

Practically these operations of addition and subtraction are restricted to notation in powers of ten; they are not much used and the above illustrations will serve as the rule for doing them.

Factoring Exponents. An exponent can be factored, one of its factors with the number going inside a parenthesis and

the other factor going outside of the parenthesis. Or if a quantity is thus expressed with a factored exponent, the parenthesis may be removed by multiplying the factors for a new exponent.

Example. Factor the exponent of 10^{33}.

Solution. $10^{33} = (10^{11})^3$ or $(10^3)^{11}$.

Example. Express the square root of 11^9 with factored exponents.

Solution. $\sqrt{11^9} = 11^{\frac{9}{2}} = (11^9)^{\frac{1}{2}}$.

Example. In the practical system of electro-magnetic units the unit of mass is taken as 10^{-11} grams, that of length as 10^9 centimeters, and that of time as 1 second. The C.G.S. unit of potential is $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$, that of resistance is LT^{-1}. Calculate the value of the practical unit of current and of resistance in C.G.S. units.

Solution. By Ohm's law the current is equal to potential divided by resistance, or $\dfrac{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}}{LT^{-1}}$. To effect the indicated division subtract the exponents of similar quantities in the denominator from those of similar quantities in the numerator. Calling the practical unit of current I and performing the division, we have $I = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$. Substituting for M and L their values as used for the practical units, we have $I = (10^{-11})^{\frac{1}{2}}\,(10^9)^{\frac{1}{2}}\,1$. It is unnecessary to repeat M, L, and T in this formula, as each is equal to unity. Thus $M \times 10^9 = 1 \times 10^9 = 10^9$. Continuing the operation, $(10^9)^{\frac{1}{2}} \times (10^{-11})^{\frac{1}{2}} = 10^{\frac{9}{2}} \times 10^{-\frac{11}{2}} = 10^{-\frac{2}{2}} = 10^{-1} = \frac{1}{10}$. This has to be multiplied by 1^{-1}. As the quantities are dissimilar the exponents cannot be added; the indicated operation may be performed. $10^{-1} \times 1^{-1} = 10^{-1} \div 1 = \frac{1}{10}$. Therefore the ampere is one-tenth of a C.G.S. unit of current.

By Ohm's law resistance is equal to the potential divided by the current, or $\dfrac{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}}{M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}} = LT^{-1}$. This result is obtained by

the process of exponential division, namely subtracting algebraically the exponents of identical quantities. Substituting for L its value, 10^9, and remembering that 1 has the same value whatever its exponent, we have, for the numerical value of resistance, $10^9 \times 1^{-1} = 10^9$ C.G.S. units.

These operations are given to illustrate the application and operations with powers of ten and with fractional exponents. The significance will be developed later, when they may be recurred to with advantage.

CHAPTER III.

MECHANICS AND PHYSICS.

Units, Simple and Compound. — Significance of Multiplication and Division of Different Kinds of Units. — Centimeter-gram-second System. — Mass. — Weight and Gravitation. — Space. — Rate Units. — Velocity. — Acceleration. — Force. — Dyne. — Gravity. — Weight. — Acceleration of Gravitation. — Energy. — Available Energy. — Entropy. — Potential Energy. — Kinetic Energy. — Varieties of Energy. — The Erg. — Power or Activity. — British System of Units. — Value of Kinetic Energy. — Equivalence of Units. — The Volt-Coulomb. — The Joule. — The Watt. — Heat Energy. — Relations of Different Units. — Relation of Power Units and Energy Units. — Efficiency. — Central or Radiant Force. — Force of a Plane on a Point near its Surface. — Theory of Dimensions. — Dimensions of Mechanical Units.

Units, Simple and Compound. Units are the basis of measurement. Thus a foot or an inch is a unit of length; a gram or a pound is a unit of weight. For different things there are different units; electric current, mechanical energy and heat, for example, have each their own units.

The same thing may be measured in some cases by different systems of units. Thus length may be measured by the English system, the inch, the foot, and other units being employed, or by the metric system, the centimeter, the meter, and other units being employed, or by any one of the many systems used by different nations at different epochs. There are two systems of electric units — the electro-magnetic and the electro-static systems. Most calculations in electric engineering are in electro-magnetic units.

A simple unit is one into which only one factor enters. The centimeter and the pound are examples of simple units.

A compound unit is composed of two or more simple units. The metric system unit of velocity is a compound unit; it is one centimeter per second. A horse-power in the English

system is expressed as the power which can raise 33,000 pounds one foot in a minute. This quantity is expressed by a compound unit; it is 550 foot-pounds per second and three simple units make it up, the foot, the pound, and the second.

A compound unit, expressed by units with hyphens between them, implies multiplication of one by the other.

Example. 7,200 foot-pounds of energy are expended in lifting a weight 109 feet. What is the weight?

Solution. The hyphen indicates that in the compound unit "foot-pounds" the feet are multiplied by the pounds. Therefore 109 feet must be multiplied by such a number of pounds as will give a product of 7,200. This number is 66.06, which is the number of pounds lifted. The product of 109 feet by 66.06 pounds gives 7,200 foot-pounds.

When the words "in a," the word "per," "in," or "a" stand between units, the division of the coefficients of the units into one another is implied. These are rate units.

Example. A trolley car is timed over a distance of 10 rails; each rail is 60 feet long. It covers the distance in 15 seconds. How many feet in a second does it travel?

Solution. The car travels 600 feet per 15 seconds. The rate per second is obtained by dividing the coefficient of feet, which is 600, by the coefficient of seconds, which is 15. $600 \div 15 = 40$, the number of feet per second.

Significance of Multiplication and Division of Different Kinds of Units. The constituent units of a compound unit, written with a hyphen, practically never have any coefficient of either of the constituent units expressed, unity being the coefficient. A gram-centimeter is one gram multiplied by one centimeter. 10 gram-centimeters are ten times the above. The coefficient, 10, applies to and multiplies the compound unit, gram-centimeter.

In the use of units the meanings of the words "multiplication"

and "division" are extended so as to include the practical multiplication and division of different classes of units with each other. Thus a foot-pound is taken as a multiplication of a foot by a pound. 150 foot-pounds is taken as the multiplication of 10 pounds by 15 feet, or as the multiplication of 15 pounds by 10 feet, all of which are impossibilities.

The difficulty may be met thus. Suppose 10 feet are to be multiplied by 15 pounds. 10 and 15 may be treated as coefficients of feet and pounds, and the multiplication may be expressed as $10 \times 15 = 150$ compound units, each compound unit being a foot-pound.

Centimeter-Gram-Second System. The scientific bases of units are the centimeter, gram, and second. On these a whole series of units electrical, mechanical, and others have been founded. The system is called the C.G.S. system.

The centimeter is the unit of length. It is approximately four-tenths of an inch, or one-thirtieth of a foot.

The gram is the unit of mass. It is the quantity of matter contained in a gram. Its relation to weight is abstracted from its status as a unit of mass. A gram weighs approximately fifteen grains (15.432+).

The second is the unit of time.

Mass. Mass is quantity of matter. In a given portion of matter it is invariable, and is unaffected by the relations of the portion of matter to the rest of the universe. The mass of a gram would be the same at any place, — at the center of the earth, on the surface of the earth, on the surface of a planet, or elsewhere.

Weight and Gravitation. Weight is mass acted on by gravity. The weight of a given portion of matter depends upon the intensity of the force of gravity. As this force varies in intensity at different parts of the universe, the weight of a given portion of matter varies also. In the center of the earth there

would be no weight. On the surface of the planet Jupiter the weight of the mass of a gram or other quantity would be much greater than on the earth. Weight is greater at the poles of the earth than at the equator; a pound would weigh less at the equator than in the polar regions, principally on account of centrifugal force, partly on account of the difference between the equatorial and polar diameters of the earth.

The force of gravity of the earth is called its gravitation. As the earth is indefinitely large with respect to the masses of engineering, the gram, pound, ton, and others, it acts upon such masses with a force proportional to their masses. It therefore tends to impart to any such mass the same velocity after acting on it for a second. This velocity is about 981 centimeters per second and varies with the latitude.

Space. Space may be of one, two, or three dimensions. Space of one dimension is length, that of two dimensions is area, and that of three dimensions is volume. Generally when space is spoken of in this book it is space of one dimension or length that is referred to.

Rate Units. One class of units expresses rate. If a current of electricity flows at the rate of 1 coulomb per second, it flows at the rate of 1 ampere. If electrical energy is expended or developed at the rate of 10 megergs (see page 24) per second, the unit expressing this rate is the watt.

Example. A generator delivers 8,050 coulombs in an hour. What is the rate?

Solution. There are 3,600 seconds in an hour. Dividing the total amount of electricity delivered by the time in seconds gives the rate per second. $8{,}050 \div 3{,}600 = 2.24$ amperes.

Example. What quantity of electricity will be delivered by an amperage of 13 in 5 minutes?

Solution. 13 amperes of current is a rate of 13 coulombs in a second. In 5 minutes there are 300 seconds. Therefore

the current will pass $13 \times 300 = 3{,}900$ coulombs in 5 minutes.

Sometimes a unit simple in form is so used as to imply rate. Thus when a velocity of 2 or any other number of length units is spoken of, what is meant is a rate of so many of the units per second.

Velocity. Velocity is lineal space traversed per second. A velocity of 10 feet is 10 feet traversed per second.

Example. A train of 60-foot cars is timed as it passes an observer. It requires 4 seconds for 6 cars to pass. Calculate the velocity.

Solution. 6 cars each 60 feet long have a total length of 360 feet. The space traversed in a second is equal to $360 \div 4 = 90$ feet. The velocity of the train is 90 feet.

Acceleration. Change of velocity per second, which is the rate of change of velocity, is acceleration. If it is an increase of velocity it is positive; if it is a decrease of velocity it is negative. The word "positive" is not generally expressed but is understood; the word "negative" must be expressed when the acceleration is of that sign.

Example. A body falls in a vacuum, starting from rest. At the end of 5 seconds it has a velocity of 4,905 centimeters. What is the acceleration, it being understood that the change in velocity has been constant and uniform?

Solution. The change of velocity per second is found by dividing the total change by the time required. This is

$$4{,}905 \div 5 = 981.$$

The acceleration is 981 centimeters.

Example. A ball is thrown into the air. It starts with a velocity of 96.6 feet and ceases to rise after 3 seconds. What is its acceleration?

Solution. Proceeding as before, we find

$$96.6 \div 3 = 32.2.$$

The acceleration is numerically 32.2 feet; but as it is a decreasing acceleration it is negative, or -32.2.

The last three problems refer to rate units, so that they are solved by division of the first by the second quantity. The quantities are respectively 360 feet per or in 4 seconds, 4,905 centimeters in 5 seconds, and 96 feet in 3 seconds.

The C.G.S. units of velocity and acceleration are 1 centimeter per second.

Force. Force is that which can change the state of motion or rest of a mass by acting on it for a period of time. It is that which can impart velocity to a mass by acting on it for a period of time.

The Dyne. The C.G.S. unit of force is the dyne. It is the force which in one second can produce an acceleration of one centimeter in a mass of one gram; the force which can impart unit velocity in unit time to unit mass. The acceleration acquired in a second by a mass of a gram is the measure of the force acting on it, it being assumed that it is perfectly free to move and that its inertia is the only resistance it opposes to motion.

Example. A mass of 1,107 grams has a velocity of 1,471 centimeters imparted to it in $1\frac{1}{2}$ seconds. How many dynes have acted on it?

Solution. The acceleration is $1{,}471 \div 1\frac{1}{2} = 980\frac{2}{3}$ centimeters. If it were one gram the force would be $980\frac{2}{3}$ dynes. As it is 1,107 grams, the force is $1{,}107 \times 980\frac{2}{3} = 1{,}085{,}598$ dynes.

The word "force" is by usage sometimes applied inaccurately. Electro-motive force is not really a form of force and cannot be measured by force units.

Gravity. Gravity is a true force. The gravity of the earth can impart in one second a velocity of 981 (about) centimeters to a mass. In other words it acts upon a gram with a force of 981 dynes. As the earth's gravity acts upon masses in proportion

to their masses, it can impart this velocity to a mass of any dimension.

Weight. The action of the earth's gravity on mass is weight. A gram is a measure of weight just as it is a measure of mass. Force is often expressed in weight units, such as the pound, gram, or ton. Weight units of force are inexact, because the attraction of the earth varies with locality.

Example. How many dynes does a kilogram weight represent ?

Solution. A kilogram is equal to 1,000 grams. The force of gravity acts upon a gram with a force of 981 dynes. Therefore it acts upon a kilogram with a force of $1,000 \times 981 = 981,000$ dynes.

As there are 1,000 milligrams in a gram, a dyne is approximately represented by $\frac{1,000}{981} = 1.02$ mg. (milligrams).

Acceleration of Gravitation. The force of gravity is measured by the acceleration it can impart to masses of matter. The gravitation of the earth varies from 978.1 to 983 centimeters, which is its acceleration and which can be expressed in any other unit of length, as in feet or inches. Thus the acceleration of terrestrial gravity is about 32.2 feet.

There are therefore two kinds of force units; one kind is based on inertia and in the C.G.S. system is the dyne; the other is based on the earth's gravitation and in the C.G.S. system is the gram. One class is that of inertia units; the other is that of gravity units.

Gravity acts on unit mass with a force numerically equal to the acceleration it imparts, which as we have seen is the velocity imparted in a second, about 981 centimeters.

To reduce inertia units of force to gravity units divide by the acceleration of gravitation; in the C.G.S. system divide by 981. To reduce gravity units to inertia units multiply by the same figure.

Example. What force is required to overcome the inertia of a mass of 1,000 kilograms so as to impart to it a velocity of 9 kilometers per hour in 5 seconds?

Solution. 1,000 kilograms = 1,000,000 grams. 9 kilograms per hour = 900,000 centimeters per 3,600 seconds = 250 centimeters per second. This velocity is acquired in 5 seconds, giving an acceleration of 250 ÷ 5 = 50 centimeters. Multiplying the acceleration by the mass it is imparted to gives the dynes of force, or 1,000,000 × 50 = 50,000,000 dynes. Dividing this by the acceleration gives the gravity units of force, or 50,000,000 ÷ 981 = 50,968 grams of force.

Energy. Energy is the overcoming or capability of overcoming a resisting force along a lineal space traversed. It is the product of force by space traversed by the point of application of the force. It is therefore measured in two kinds of units, inertia and gravity units, just as force is measured, and the reduction of one to the other is effected by division or multiplication by the acceleration of gravitation exactly as in the case of force units.

Energy may be expended or developed; neither operation can occur alone; both must be simultaneous and equal in amount. If a given amount of energy is expended, an exactly equal amount of energy is developed. This is the doctrine of the conservation of energy.

Available Energy.—Entropy. Man's economic needs are served by transformation of energy, generally by the transformation of higher grade energy into lower grade. If all energy were of the same grade none would be economically available for the uses of mankind. Available energy is called entropy. The available energy of the universe is constantly diminishing.

Coal and the oxygen of the air existing separately represent high grade energy. When coal is burned the two combine,

producing heat energy. A small part may be utilized by man. After burning the grade of the remaining and unutilized energy is so low that it cannot be used. Entropy has been lost, but energy remains unchanged.

Potential Energy. Potential energy is the power of exerting energy, which may be present in inert matter owing to circumstances of position or other states. A weight raised to a height has the capability of exerting energy in its descent to its original position. This capability is potential energy. The separate existence of coal and oxygen represents potential energy. In burning the two combine; the potential energy is converted into active energy, which is heat in this case.

Kinetic Energy. Energy due to motion is called kinetic energy. A cannon ball in motion can exert energy, as in piercing an armor plate, which energy is due to its motion and is kinetic energy.

Varieties of Energy. There are many kinds of energy, such as mechanical, heat, chemical, and electric energy.

The Erg. The C.G.S. unit of energy is the erg. It is the energy exerted by a force of one dyne acting along a path one centimeter long and acting in the direction of its motion. The point of application of the force moves along the path. It is an inertia unit. One million ergs are a megerg.

The erg is an inconveniently small unit in many cases, and in such values as the above the megerg is used. To convert ergs into megergs, move the decimal point 6 places towards the left.

Example. How many ergs are exerted in raising 15 kilograms a distance of 3 meters?

Solution. 15 kilograms = 15,000 grams. 3 meters = 300 centimeters. Multiplying these gives the energy in gravity units; 15,000 × 300 = 4,500,000 gram-centimeters. Multiplying this by the acceleration of gravitation gives the equiva-

lent in inertia units, or ergs. $4{,}500{,}000 \times 981 = 4{,}414{,}500{,}000$ ergs $= 4{,}414.5$ megergs.

Power or Activity. Rate of energy is termed power or activity. Power units can be reduced to ergs per second or to other energy units per second. Power or activity is the rate of expending, developing, or transforming energy.

Example. In the last example what was the power exerted if the 15 kilograms was raised the 3 meters in 6 seconds?

Solution. Since power is ergs per second in the C.G.S. system, to get the answer in that system for inertia units divide the energy by the time. $4{,}414.5 \div 6 = 735.75$ megergs per second.

Example. 27 dynes act along a path 27 centimeters long. What is the energy in ergs?

Solution. $27 \times 27 = 729$ ergs.

Example. If 729 ergs are exerted in overcoming a resisting force along a space traversed of 25 centimeters' length, what is the force?

Solution. $729 \div 25 = 29.16$ dynes.

British System of Units. The British system of units is founded on three fundamental units, the foot, the pound, and the second.

The unit of velocity is a rate of 1 foot per second.

The unit of acceleration is the velocity acquired in 1 second. It is obtained in any given case by dividing the velocity acquired by the time required to attain it.

Example. In 13 seconds a car attains a velocity of 7 feet per second. What is its acceleration?

Solution. It is $7 \div 13 = 0.538$ foot.

The inertia unit of force in this system is the poundal. It is the force which can impart a velocity of 1 foot per second to 1 pound by acting on it for 1 second.

Example. If in the last example the car weighed 40,000 pounds what force was exerted on it in poundals?

Solution. As a velocity of 0.538 foot was imparted to 40,000 pounds in 1 second, the poundals of force were $0.538 \times 40{,}000 = 21{,}520$ poundals.

The gravity unit of force is the pound avoirdupois. Gravity as a force produces an acceleration of about 32.2 feet per second (which corresponds to and is equal to 981 centimeters). Therefore all gravity units of the British system may be reduced to inertia units by multiplying by 32.2.

Example. How many poundals in 6 pounds weight?

Solution. $6 \times 32.2 = 193.2$ poundals.

Inertia units of the British system are reduced to gravity units by dividing by the same figure.

The inertia unit of energy is the foot-poundal; it is the force of a poundal exerted along a path of 1 foot. The gravity unit is the foot-pound, which is the energy required to raise 1 pound a height of 1 foot.

The British unit of power is the horse-power. It is the rate of energy required to exert 550 foot-pounds of energy per second, or 33,000 foot-pounds per minute.

Example. Returning to the example on page 25, what was the force exerted in pounds, and what energy was expended in 10 feet of the car's progress?

Solution. As the poundals are in the English system, use the acceleration of gravity in feet, 32.2. Then $21{,}520 \div 32.2 = 668.3$ pounds. The energy is force multiplied by space traversed, or $21{,}520 \times 10 = 215{,}200$ foot-poundals, or $668.3 \times 10 = 6{,}683$ foot-pounds.

Example. How many foot-poundals are exerted by a 150-pound man in going up a flight of stairs 10 feet high?

Solution. 150 pounds $\times$ 10 feet $= 1{,}500$ foot-pounds, and $1{,}500 \times 32.2 = 48{,}300$ foot-poundals.

Value of Kinetic Energy. If a force a acts upon a body free to move, it will impart to it a uniformly increasing velocity, and at the end of a time t the velocity will be $\frac{l}{t}$, l indicating lineal space. This is acquired velocity. The average velocity will be one-half of this, or $\frac{1}{2} \times \frac{l}{t}$. The space traversed will be the product of the average velocity by the time required to traverse it; this product is $\frac{1}{2} \times \frac{l}{t} \times t = \frac{l}{2}$. The value of the force acting on the body is the product of the acceleration it imparts to the body multiplied by the mass, which we will call m. Acceleration is the quotient of velocity divided by the time required to attain it, or $\frac{l}{t} \div t = \frac{l}{t^2}$, and the force is $\frac{ml}{t^2}$. Energy is equal to the product of force by the space along which it is exerted; this product is then $\frac{ml}{t^2} \times \frac{l}{2} = \frac{1}{2} \frac{ml^2}{t^2} = \frac{1}{2} mv^2$.

The above formula gives the value of kinetic energy in inertia units, such as ergs or foot-poundals. To reduce these to gravity units the value given by the formula must be divided by the acceleration of gravity in the system to which the formula has been applied.

Example. Calculate the energy in a mass of 11 grams moving at the rate of 210 centimeters per second.

Solution. By the formula we have energy $= \frac{1}{2} \times 11 \times 210^2 = 242{,}550$ ergs, which divided by the acceleration of gravity in the metric unit, 981 centimeters, gives the value in the gravity system as 247 centimeter-grams, or 247 grams raised to a height of 1 centimeter against the attraction of gravity.

Example. What is the energy in a 150-pound projectile moving at the rate of 2,000 feet per second?

Solution. $\frac{1}{2} \times 150 \times \overline{2{,}000}^2 = 3 \times 10^8$ foot-poundals, or dividing by $32.2 = 9{,}316{,}770$ foot-pounds.

Equivalence of Units. The equivalents given below are used to change from one system to the other.

1 foot = 30.48 centimeters and 1 centimeter = .0328 foot.

1 inch = 2.54 centimeters and 1 centimeter = .3937 inch.

1 pound = 453.6 grams and 1 gram = .0022 pound or 15.432 grains.

From the above data:

Example. Calculate the value of the poundal in dynes.

Solution. 30.48 (centimeters) × 453.6 (grams) = 13,826 dynes.

Example. Calculate the value of the pound in dynes.

Solution. 453.6 (grams) × 981 (centimeters) = 4.45×10^5 dynes.

Example. Calculate the value of the foot-poundal in ergs.

Solution. 453.6 × 30.48 × 30.48 = 421,408+ ergs. To find the number of ergs in a foot-pound multiply 421,408 ergs × 32.2 = 13,569,338 = 13.57 megergs.

Example. How many ergs are there in a horse-power acting for one second?

Solution. Take 13.57 megergs as the value of the foot-pound. Then we have 1 horse-power-second = 550 foot-pounds = 550 × 13.57 = 7,464 megergs = 7,464,000,000 ergs.

Example. What energy in ergs is exerted in raising 80 pounds to a height of 5 feet?

Solution. 80 × 5 = 400 foot-pounds. Using the factor or equivalent determined in previous problem we have

$$13{,}569{,}338 \times 400 = 5{,}427{,}735{,}200 \text{ ergs.}$$

The Volt-Coulomb. The practical unit of electric energy is the volt-coulomb. This is based upon the centimeter, gram, and second, being a C.G.S. system unit. It is equal to 10^7 C.G.S. units of energy, which units are ergs. It is a watt-second, and is equal to a joule.

Example. Calculate the equivalent of a volt-coulomb in foot-pounds.

Solution. The volt-coulomb is equal to 10^7 ergs, which are 10 megergs. The foot-pound is equal to 13.57 megergs. Therefore a volt-coulomb is equal to $\frac{10}{13.57} = 0.737$ foot-pound.

The Joule. Another unit of energy is the joule. It is equal to 10^7 ergs, 10 megergs, and therefore to the volt-coulomb, and in many cases can be used as a synonym of the volt-coulomb.

The Watt. The practical unit of electric power is the volt-ampere, which is equal to 1 volt-coulomb per second, which is equal to 1 joule per second. This unit is called the watt. As the volt is equal to 10^8 C.G.S. units, and as the ampere is equal to 10^{-1} C.G.S. units, it follows that the volt-ampere or watt is equal to $10^8 \times 10^{-1} = 10^7$ C.G.S. units of power, which C.G.S. unit is 1 erg per second. The watt is equal to 10^7 ergs or 1 joule per second.

Example. What is the value in watts of 10^{11} ergs per minute?

Solution. $10^{11} \div 60 = 1{,}666 \times 10^6$ ergs per second. Dividing this by 10^7 gives the watts of power.

$$1{,}666 \times 10^6 \div 10^7 = 166.7 \text{ watts.}$$

Heat Energy. Heat is a form of energy. As it is due to motion of the particles of matter, it is kinetic energy. The C.G.S. unit of heat energy is the erg. It is sometimes called the primary unit of heat.

Units of Heat Energy. The engineering and working units of heat energy are based on the heat required to impart a definite increment of temperature to a definite weight of water. The increment or increase of temperature is defined by reference to one of the thermometric scales. As these are arbitrary, such units have an arbitrary relation to the C.G.S. system. Their values are determined by experiment and are not absolutely accurate.

The heat required to raise the temperature of a gram of water

1° C. is the therm, gram-degree, minor calorie, or simply calorie. It is sometimes called the secondary unit of heat. It is equal to 41.66 megergs, often given as 42 megergs. 1000 calories is the heat required to raise the temperature of a kilogram of water 1° C. This is the kilogram-degree or major calorie. It is equal to 41,666 megergs.

The heat required to raise the temperature of a pound of water 1° F. is the British thermal unit, often written B.T.U. It is equal to 778 foot-pounds, approximately.

Example. Calculate the value of the calorie in volt-coulombs.

Solution. 1 volt-coulomb = 10 megergs. 1 calorie = 41.66 megergs. Therefore 1 calorie = 4.166 volt-coulombs.

Example. Calculate the value of the B.T.U. in volt-coulombs.

Solution. 1 volt-coulomb = 0.735 foot-pound. 1 B.T.U. = 778 foot-pounds. Therefore 1 B.T.U. = 778 ÷ 0.735 = 1,058 volt-coulombs.

Energy Units and Equivalents. Energy equivalents are approximate once the C.G.S. system is departed from. Thus the value of the B.T.U. is variously given, ranging from 772 to 778 foot-pounds. The difficulties attending the accurate determination of this equivalent account for the discrepancies.

Relations of Different Units. The scope and meaning of equivalents of units may be illustrated by the watt-second. This unit of electric energy is equal in value (*a*) to 10^7 ergs, (*b*) to 1 joule, (*c*) to 0.24 calorie, (*d*) to 10,193 gram-centimeters, (*e*) to 0.00095 B.T.U., (*f*) to 0.737 foot-pound; and to others.

The watt-second is a compound unit, for which the joule is often a synonym, and is equal to a rate of energy of one watt exerted for one second. Following up the equivalents given above we see that it can (*a*) and (*b*) impart to a gram a velocity of 10^7 centimeters; (*c*) it can heat 0.24 gram of water

1° C.; (*d*) it can raise 10,193 grams to a height of 1 centimeter; (*e*) it can heat 0.00095 pound water 1° F.; (*f*) it can raise 0.737 pound to a height of 1 foot.

The relation of the C.G.S. units to each other is purely decimal. The relations of the so-called engineering units to each other are fixed and arbitrary; they are arbitrary in the sense that the values of the engineering units were not based on any system of uniform relationship.

Example. If 18 ounces of water are raised 5° F. in temperature, how many foot-pounds of energy will be absorbed?

Solution. As there are 16 ounces in a pound, 18 ounces = 1.125 pounds. Multiplying this weight by the number of degrees through which it is raised gives the B.T.U.'s developed. $1.125 \times 5 = 5.625$ B.T.U.'s. Applying the equivalent, 778, gives $5.625 \times 778 = 4{,}376.25$ foot-pounds.

Relation of Power Units and Energy Units. A power unit followed by a time unit with a hyphen intervening gives energy. Thus a horse-power-second is equal to 550 foot-pounds of energy. A watt-second is equal to a volt-coulomb, a unit of electric energy. Sometimes the hyphen is omitted.

Example. Calculate the energy in 29 horse-power-minutes.

Solution. There are 60 seconds in a minute. Therefore 29 horse-power-minutes are equal to $29 \times 60 = 1{,}740$ horse-power-seconds. A horse-power-second is equal to 550 foot-pounds. 1,740 horse-power-seconds $= 1{,}740 \times 550 = 957{,}000$ foot-pounds of energy.

Example. Calculate the electric and mechanical energy in 5,040 watts acting for 20 minutes.

Solution. 20 minutes $= 20 \times 60 = 1{,}200$ seconds. The energy of the above watt-minutes is $5{,}040 \times 1{,}200 = 6{,}048{,}000$ watt-seconds, or volt-coulombs of electric energy. The volt-coulomb is equal to the joule, so we may read for the volt-coulombs 6,048,000 joules of mechanical energy.

Example. A man runs up a flight of stairs 12 feet high in 5 seconds. He weighs 150 pounds. What power does he exert in horse-power and in watts?

Solution. In 5 seconds he expends $150 \times 12 = 1{,}800$ foot-pounds of energy; the rate per second is therefore $1{,}800 \div 5 = 360$ foot-pounds, which is equal to $360 \div 550 = 0.655$ horse-power, because a horse-power is a rate per second of 550 foot-pounds. Take the foot-pound as equal to 13.57 megergs. Multiplying this by 360 gives $360 \times 13.57 = 4{,}885.2$ megergs, or 488.5 watts, because the watt is equal to 10 megergs, or to 10^7 ergs.

Example. Taking a pound as 453.6 grams, and the foot as 30.48 centimeters, calculate the equivalent of the horse-power in watts using 981 as acceleration of gravity.

Solution. A horse-power is 550 foot-pounds per second. 1 foot-pound is equal to $30.48 \times 453.6 = 13{,}825.728$ gram-centimeters. Taking the acceleration of gravity as 981 centimeters, the last figure multiplied by 981 gives dyne-centimeters, or $13{,}825.728 \times 981 = 13{,}563{,}039$ ergs, because an erg is a dyne-centimeter. A watt is 10 megergs per second; a foot-pound, therefore, in watts is given by dividing the value in ergs by 10^7, giving 1.356 watts. The horse-power is equal to 550 foot-pounds, or in watts to $1.356 \times 550 = 746$ watts.

This problem could be solved by simply multiplying 550 (foot-pounds per second) by 13.57 and then dividing by 10.

Efficiency. The relation of useful to total energy in any process is called efficiency. Useless energy is that which is expended in overcoming friction and hurtful resistances generally. Useless energy is always developed at the same time with useful, and is manifested in the heating of bearing surfaces, of electric conductors, and in other ways.

Efficiency is the ratio of the part of the energy utilized to the total energy expended. It is generally stated as a percentage.

Example. 7 megergs are expended each second in driving a dynamo. 6 megergs per second are delivered to the system to be there utilized. What is the efficiency?

Solution. $6 \div 7 = 0.857$ is the efficiency decimally expressed. It is 85.7 per cent. The remainder, 14.3 per cent, is wasted.

Central or Radiant Force. If two points at a distance from each other attract or repel each other, the force exerted upon each one will vary inversely as the square of the distance.

Example. Assume that two magnet poles attract each other with a force of 9 dynes when $2\frac{1}{2}$ centimeters apart. The distance is increased to 3 centimeters. What will the attraction be at this distance?

Solution. The proportion of the inverse squares of the distances is $\overline{3}^2 : \overline{2\frac{1}{2}}^2 : : 9 : x = 5.0625$ dynes.

The attraction could have been expressed in other units, such as grains or grams.

The attraction or repulsion exerted by two points upon each other varies with the product of the force of one body by that of the other.

Example. Assume that there are two electrically charged pith-balls at such a distance from each other that they have with respect to each other 2 units of force and 3 units of force respectively. With what force will they attract each other?

Solution. The product of their individual attractions is $3 \times 2 = 6$ units of force. The force may be measured in any convenient unit, such as dynes.

Distance and the forces exerted by both points enter into the problem simultaneously in many cases.

Example. One north magnet pole has 5 megergs force as regards its action on the south pole of another magnet situated 3 centimeters distant from it. The south pole of the other magnet has 7 megergs force under identical conditions. Cal-

culate their mutual attraction at the stated distance of 3 centimeters and at 4 centimeters.

Solution. Their mutual attraction is equal to the product of their individual attractions, or

$$5 \times 7 = 35 \text{ megergs.}$$

This is at the distance of 3 centimeters. For their attraction at the distance of 4 centimeters we must apply the rule of the inverse squares, thus:

$$4^2 : 3^2 :: 35 : x = 19.7 \text{ megergs.}$$

Example. At 1 centimeter distance the individual forces of two magnet poles are 9 and 17 dynes respectively. Calculate the combined attraction at the distance of 19 centimeters.

Solution. The mutual attraction of the poles at the stated distance of 1 centimeter is

$$9 \times 17 = 153 \text{ dynes.}$$

The attraction at the distance of 19 centimeters may be obtained by the rule of the inverse squares. As the one term of the proportion is 1, or unity, it is sufficient to divide the attraction at 1 centimeter by the square of the other distance, thus:

$$153 \div \overline{19}^2 = 0.424 \text{ dyne.}$$

For the above rules to hold the poles or other things acting on each other must be small compared to the distance. The law then applies with reasonable accuracy. To be rigorously accurate the points acting on each other must be infinitely small.

Force of a Plane on a Point near its Surface. A plane of indefinitely large size exerting force on a point does so with a force numerically expressed as $2\pi\sigma m$, in which σ (the Greek letter sigma) indicates the force per unit area of the plane and m the force of the point. The point might be a magnet pole and the plane the face of a magnet, or it might be a mass acted on by gravity. The force is the same irrespective of distance,

provided the area of the plane is large enough. The law is deduced by calculus.

Theory of Dimensions. By the use of what is known as the theory of dimensions the relations of mechanical and physical quantities to each other are expressed in algebraic expressions into which only three quantities enter, namely, space, time, and mass. They were an invention of Fourier, and Clerk Maxwell brought them into prominence. They are not of frequent use in engineering work, but are of great value in the theoretical aspect, and as they are really very simple, should be studied.

Exponential notation is used in the discussion of dimensions.

Dimensions of Mechanical Units. Units in the absolute and in the practical systems of mechanical and electric units are derived directly from the three fundamental units of length, mass, and time, namely, the centimeter, the gram, and the second. These fundamental units are designated by the letters L, M, and T. Mechanical units will first be treated.

Velocity is equal to the length traversed in a given time divided by the time required to traverse the length in question. Its dimensions are L/T, or LT^{-1}.

Acceleration is the rate of change of velocity and is equal to the velocity acquired in a given time divided by the time required to attain such velocity. Its dimensions are therefore velocity divided by time, $L/T \div T$, or LT^{-2}.

Example. A trolley car starts from rest and in 12 seconds is moving at the rate of 200 feet in 21 seconds. What is the velocity attained and what is the acceleration?

Solution. Substituting for L and T their values gives

$$\text{Velocity} = LT^{-1} = 200 \div 21 = 9.524 \text{ feet, or } 290 \text{ cm.}$$

$$\text{Acceleration} = \dot{L}T^{-1} \times T^{-1} = 9.52 \div 12 = 0.7936 \text{ feet, or } 24.19 \text{ cm.}$$

Force is measured by the acceleration it can impart per unit

of mass. Its dimension is the product of mass by acceleration. It is clear that it takes more force to give acceleration of a given amount as the mass is larger. It takes twice the force to impart a given velocity to a mass of two grams that it does for a mass of one gram. Hence the multiplication is needed to express the force required to give acceleration to a mass, and the product of the multiplication is the number of force units required. Its dimension is the product of mass M by acceleration LT^{-2}; $M \times LT^{-2} = MLT^{-2}$ are the dimensions of force.

Example. How many dynes were required to impart the velocity of the last problem, assuming the car to weigh 40,000 pounds?

Solution. The mass of the car is $40{,}000 \times 453.6 = 18{,}144 \times 10^3$ grams. The acceleration is 24.19 cm. Multiplying mass by acceleration gives $18{,}144 \times 10^3 \times 24.19 = 439 \times 10^6$ (about) dynes.

This can be done by use of T and T' for the last example. Force $= MLT^{-1}T'^{-1} = 18{,}144 \times 10^3 \times 6{,}096 \div (21 \times 12) = 439 \times 10^6$ (about) dynes.

The factor 6,096 is the equivalent of 200 feet in centimeters.

It will be observed that in this example there are two values of T, which have been indicated by T and T'. In other words, to use the dimensions directly it is necessary to take cognizance of the two values, as is done in the solution. Where two values of the same unit occur it is often possible to modify the dimensional formula so as to substitute for the square of L, M, or S the product or rectangle of two such symbols, distinguished by prime marks or other distinguishing characters.

Example. Apply the method as above to the first example.

Solution. As there are two values of T, the formula can be modified to read $LT^{-1}T'^{-1}$. Substituting for the symbols their values as given in the statement of the problem, we have

$$\text{Acceleration} = 200 \div (21 \times 12) = \frac{200}{252} = 0.7936 \text{ feet, as before.}$$

This treatment is merely given as a process by which dimensional formulas can be directly applied to the solution of problems. Strictly speaking they should be regarded as expressing proportional relations of quantities.

If a force acts along a path it must necessarily overcome a resistance and its point of application moves along. The exercise of force along a path, or along a length traversed, is energy. Its dimensions are the product of force MLT^{-2} by space L, which gives ML^2T^{-2}. As constant factors do not enter into dimensions, a factor, $\frac{1}{2}$, which is required to make this expression numerically correct in calculations, is omitted. In the numerical sense the expression for energy is $\frac{1}{2} ML^2T^{-2}$. By using the rectangle of factors instead of their squares, calculations can be made by simple substitution.

Example. A force acts upon a mass of 200 grams and in 5 seconds moves the mass a distance of 750 cm. What energy is expended upon the mass? It is understood that the mass derives all this motion from the force and that the force is uniform.

Solution. Let L represent the distance traversed by the body and let L' represent the distance it would traverse in 5 seconds at the velocity imparted by the force. L' is equal to $2L$, because for the purposes of the problem the mass may be taken as starting from rest, and it receives a perfectly uniform increase of velocity. The distance traversed in the 5 seconds, which is 750 cm., is equal to the average of the starting and terminal speeds, referred to 5 seconds. The starting speed is 0; the terminal speed is L'; the average is $(0 + L') \div 2 = L'/2 = 750$; whence $L' = 1{,}500$. Substituting in the formula, energy $= ML \times L' \div T^2$ gives

$$\text{Energy} = (200 \times 750 \times 1{,}500) \div 5^2 = 9{,}000{,}000 \text{ ergs.}$$

Power is the rate of expenditure of energy. Its dimensions are the quotient of energy ML^2T^{-2} divided by time T, giving ML^2T^{-3}.

Example. What power was expended upon the mass in the last example?

Solution. The 9×10^6 ergs of energy were expended in T seconds, the power was $9 \times 10^6 \div 5 = 18 \times 10^5$ C.G.S. units of power. As a watt is 10^7 C.G.S. units, the power is 0.18 watt.

Momentum is the quantity of motion in a body, as often defined. It is mass M multiplied by the velocity L/T at which it moves, giving MLT^{-1}. It is momentum to which Newton's laws of motion refer.

PROBLEMS.

Reduce 60 feet in 3 seconds to feet per second.
Ans. 20 feet per second.

Reduce 5,280 feet per minute to feet per second.
Ans. 88 feet per second.

A train runs at 64 miles an hour; how many feet per second is its speed? *Ans.* 93.9 feet per second.

A trolley car is timed over a distance of ten rails; each rail is 60 feet long, and it covers the distance in 16 seconds. Calculate the velocity. *Ans.* 37.5 feet.

A machine exerts 7,000,000 foot-pounds of energy per hour. Calculate the horse-power. *Ans.* 3.53 + horse-power.

A train moves at the rate of 11 car lengths in 9 seconds; each car is 60 feet long; how many C.G.S. units of velocity does this reduce to? Assume the foot to be 30.48 cm. in length.
Ans. 2,235 C.G.S. units.

What velocity is 310 feet in 59 seconds? *Ans.* 5.25 feet.

A car starts from rest and in 120 feet attains a speed of 10 miles per hour. Calculate the acceleration. *Ans.* 0.89 feet.

A mass of 21 grams has a velocity of 390 cm. imparted to it. A period of 7 seconds was required to impart this velocity. How many C.G.S. units of force were exerted? *Ans.* 1,170 dynes.

How many ergs are there in a mass of 5 grams, free to fall 1 meter, the acceleration of gravitation being 981 cm.?
Ans. 490,500 ergs of potential energy.

How many ergs are present in a mass of 397 grams moving at a speed of 2,340 cm. per minute?

Ans. 301,918.5 ergs of kinetic energy.

A mass of 5 grams has a rate of motion of 3 cm. per 6 seconds imparted to it in a period of 5 seconds. Calculate the (*a*) final velocity, (*b*) acceleration, (*c*) force, (*d*) energy present at the end of the period.

Ans. (*a*) 0.5 cm.
(*b*) 0.1 cm.
(*c*) 0.5 dyne.
(*d*) 0.625 erg.

Calculate the foot-pounds of energy in a 20-ton car moving at the velocity of 30 feet. *Ans.* 559,006

If brought to rest in 11 seconds, what power is exercised in doing this? *Ans.* 50,819 foot-pound-seconds.
92.4 horse-power.

A 12-kilowatt generator runs for 35 minutes. What are the joules?
Ans. 252×10^5 joules, or 25,200,000 joules.

How many foot-poundals are there present in a 20-ton car moving at 26 miles an hour? *Ans.* 29,083,022 foot-poundals.

If the above car is brought to rest in 30 seconds what horse-power is expended in the braking? *Ans.* 54.73 horse-power.

A generator produces a current of 50 amperes at 112 volts. Calculate the horse-power. *Ans.* 7.51 horse-power.

How many 110-volt lamps averaging one-half ampere each are to be allowed to the horse-power? *Ans.* 13.56 lamps.

If the earth and the moon were each of twice its present diameter and of the same specific gravity, how would the attraction of one for the other be affected?

Ans. As mass varies with the cube of like linear parts the mass of each would be 8 times as great as now, and the attraction would be $8 \times 8 = 64$ times the present.

A magnet pole of 2 dynes strength is 30 cm. from one of 3 dynes strength. Calculate the attraction of one for the other.

Ans. 0.0066 dyne.

To what height must 51 grams be raised to give 392 gram-centimeters? *Ans.* 7.686 cm.

If a dynamo armature is making 750 revolutions per minute, how many revolutions does it make per second? *Ans.* 12.5.

One ampere at a pressure of one volt is a volt-ampere. Express in correct form 11 amperes at 12 volts pressure.

Ans. 11 × 12 = 132 volt-amperes (not 11 volts × 12 amperes = 132 volt-amperes).

What is the rate of electric energy if a generator delivers 36,000 megergs in an hour? *Ans.* 10 megergs per second = 1 watt.

A piece of some material weighs 251 pounds on a spring balance at the equator. What will it weigh at the pole, taking the ratio of the gravitation at the two points as 983 and 978?

Ans. 252.3 pounds.

How many calories does a 150-kilowatt generator develop per hour? *Ans.* $1,296 \times 10^5$.

How many joules are there in a kilowatt-hour?

Ans. 36×10^5.

Express the last two answers in ordinary numeration.

Ans. 129,600,000 calories.
3,600,000 joules.

550 watts are passed through the coil of an electric boiling apparatus. The boiler contains one pint of water. After 6 minutes the water is brought from 60° F. to boiling (212° F.). What is the efficiency? *Ans.* 81 per cent.

(1 pint of water = 1 pound; 1,058 watt-seconds = 1 B.T.U.)

If a dynamo produces energy at the rate of 3,500 B.T.U. per hour, what is the rate of output? *Ans.* 1,025 watts.

A motor absorbs 500 watts. It drives a pump and raises in an hour 1,500 gallons of water 90 feet. What is the efficiency of the system? *Ans.* 45 per cent. (nearly)

CHAPTER IV.

OHM'S LAW.

Three Factors of an Active Circuit. — Ohm's Law. — Several Appliances in One Circuit. — Application of Ohm's Law to Portions of a Circuit. — Simple Method of Expressing Ohm's Law. — Proportional Form of Ohm's Law. — Fall of Potential. — *RI* Drop. — Counter Electro-motive Force. — Problems.

Three Factors of an Active Circuit. In an active circuit there are three factors on which its action depends. These are current, electro-motive force, and resistance. The relation of these three in a circuit through which a current is passing is embodied in Ohm's law.

Ohm's Law. It has been found by experiment that in a conductor of given resistance the intensity of current varies with the electro-motive force (abbreviated as e.m.f.). It has also been found that in conductors of different resistances the currents due to the same e.m.f. vary inversely as the resistances. When both e.m.f. and resistance vary, the relation of the current to such variations is expressed in the proportion given below, in which E, I, and R indicate e.m.f., current intensity, and resistance respectively.

Let a unit current be assumed to pass through a resistance R. For this to take place there will have to be an e.m.f. Now assume that another current of intensity I is to pass through the same resistance instead of the current 1, then the e.m.f. will have to be different, and the relation of these quantities will be given in the proportion

$$1 : I :: R : E,$$

which is a proportion expressing Ohm's law.

From the last proportion are derived the three equations or formulas

$$I = \frac{E}{R}; \qquad E = RI; \qquad R = \frac{E}{I}.$$

These are the most used expressions of Ohm's law.

In words the law is generally expressed thus:

The current intensity is equal to the electro-motive force divided by the resistance.

The electro-motive force is equal to the product of the resistance by the current.

The resistance is equal to the electro-motive force divided by the current.

Example. An electro-motive force of 5 volts is expended on forcing a current through a resistance of 10 ohms. What is the intensity of the current?

Solution. Applying the first form of the law, we divide the e.m.f. by the resistance, obtaining

Intensity of current = 5 volts ÷ 10 ohms = $\frac{1}{2}$ ampere.

Example. What e.m.f. is required to force 10 amperes through a resistance of 105 ohms?

Solution. Here the second form of the law may be employed. Multiplying the resistance by the current gives the e.m.f., or

$$105 \times 10 = 1{,}050 \text{ volts.}$$

Example. An e.m.f. of 27 volts forces a current of 5 amperes through a wire. What is the resistance of the wire?

Solution. The third form of the law applies here. Dividing the e.m.f. by the current intensity we have

$$27 \div 5 = 5\tfrac{2}{5} \text{ ohms.}$$

The different classes of problem of which the above are examples can all be done by the one form of Ohm's law. It is more convenient to apply the three forms respectively as illustrated in the three examples.

Several Appliances in One Circuit. The elements which determine the intensity of a current are the total resistance and the total e.m.f. in the circuit, irrespective of the distribution of the same therein. Assume several batteries to be dis-

tributed on a circuit. Each battery introduces two things in the system, e.m.f. and resistance, both of which have to be taken into consideration.

Example. An electric circuit has distributed along its course three sources of e.m.f. A battery of 8 ohms resistance and 2 volts e.m.f. is connected by a wire of 107 ohms resistance to a second battery of 20 ohms resistance and 21 volts e.m.f. This is connected by a wire of 1 ohm resistance to a third battery of 5 ohms resistance and 10 volts e.m.f. The circuit is closed by a wire of 17 ohms resistance, connecting the first to the third battery. All the batteries work together, being of identical polarity. Calculate the current.

Solution. The total resistance of the circuit is made up of the resistance of the batteries added to that of the conductors. This gives

$$R = 8 + 107 + 20 + 1 + 5 + 17 = 158 \text{ ohms.}$$

The total e.m.f. is the sum of the e.m.f.'s of the batteries. This gives

$$E = 2 + 21 + 10 = 33 \text{ volts.}$$

By Ohm's law we have

$$I = \frac{33 \text{ volts}}{158 \text{ ohms}} = 0.209 \text{ ampere.}$$

Application of Ohm's Law to Portions of a Circuit. Ohm's law applies to any portion of a conductor through which a current is passing. The intensity of the current passing through a circuit is equal not only to the e.m.f. of the circuit divided by the resistance thereof but is also equal to the e.m.f. expended on any portion of the circuit divided by the resistances of the same portion. This will be found treated in one of its bearings elsewhere in this book.

Let $e, e_1, \ldots e_n$ denote the e.m.f.'s expended on portions of an electric circuit, and let $r, r_1, \ldots r_n$ denote the resistances

of the respective portions corresponding thereto; then, calling the intensity of current I, as before, we have

$$I = \frac{e + e_1 + \ldots + e_n}{r + r_1 + \ldots + r_n}, \text{ and also } I = \frac{e}{r} = \frac{e_1}{r_1}, \text{ and so on.}$$

This formula expresses an almost obvious law, that while the e.m.f. expended on portions of an active circuit varies with the resistances of the same portions, the current is uniform in all parts of the circuit.

Example. A voltmeter connected to two parts of an active circuit shows 40 volts. The resistance of the wire included between the points where the voltmeter terminals are connected is $\frac{16}{17}$ ohm. What current is passing through the circuit?

Solution. We have as the values of e and r of the formula last given 40 and $\frac{16}{17}$, so that

$$I = 40 \div \frac{16}{17} = \frac{85}{2} = 42\tfrac{1}{2} \text{ amperes.}$$

Simple Method of Expressing Ohm's Law. A very ingenious way of representing and of memorizing Ohm's law is embodied in the following device:

$$\frac{E}{R \times I}.$$

If any one of the elements is removed, the relative position of the other two gives the value of the third in terms of the other two. Thus, if from the group we remove E, $R \times I$ are left; therefore the value of E in terms of R and I is $R \times I$, or the product of R and I as it should be. If R is removed from the group, E/I remains, giving the value of R in terms of E and I, which is E divided by I. In the same way, if I is removed from the group its value remains, just as in the other cases, namely, E/R, or E divided by R.

Example. The e.m.f. between the ends of a conductor is to be determined. Its resistance is 15 ohms and a current of 5 amperes is maintained through it. Apply the diagram.

Solution. Removing E from the diagram leaves $R \times I$. Substituting for R and I their values gives the value of E or of the e.m.f. as $15 \times 5 = 75$ volts.

Example. Take the e.m.f. between the ends of a conductor as 30 volts and its resistance as 15 ohms. What current will pass through it?

Solution. Removing the symbol of current I from the diagram there remains E/R, and substituting the values of resistance and e.m.f. from the statement of the problem we obtain $30/15 = 2$ amperes.

Example. Let a current of 25 amperes be maintained by an e.m.f. of 29 volts. What is the resistance of the conductor in which it is so maintained?

Solution. Removing R from the diagram leaves E/I, and substituting as before for E and I their values from the conditions as stated in the problem we find

$$R = \frac{E}{I} = \frac{29}{25} = 1.16 \text{ ohms.}$$

Proportional Form of Ohm's Law. Ohm's law can be employed in the proportional form. It is a form but little employed in practical work but is available for practice by the student. Three principal forms may be stated, corresponding to three principal forms of the law.

The resistance varies with the quotient of the electro-motive force divided by the current.

$$R_1 : R_2 :: \frac{E_1}{I_1} : \frac{E_2}{I_2}. \tag{1}$$

The current varies with the quotient of the electro-motive force divided by the resistance.

$$I_1 : I_2 :: \frac{E_1}{R_1} : \frac{E_2}{R_2}. \tag{2}$$

The electro-motive force varies with the product of the resistance by the current. $E_1 : E_2 :: R_1I_1 : R_2I_2.$ (3)

Example. Compare the resistance of two conductors with potential differences between their ends of 7 and 16 volts respectively and currents of 13 and 17 amperes respectively produced thereby.

Solution. Applying formula (1) we find

$$R_1 : R_2 : : \frac{7}{13} : \frac{16}{17} = 0.539 : 0.941.$$

This proportion gives more than the mere ratio, for the third and fourth terms of the proportion give the values of R_1 and R_2 namely, 0.539 ohm and 0.941 ohm respectively.

Example. The e.m.f. of a battery is 10.7 volts and its resistance is 50 ohms. A conductor of 1,101 ohms resistance connects the terminals of the battery. Compare the current with that which would be produced through the same conductor by a battery of 6.42 volts and of 30 ohms.

Solution. Proportion (2) gives

$$I_1 : I_2 : : \frac{10.7}{1,151} : \frac{6.42}{1,131} = 0.0093 : 0.0057.$$

Again we have the result expressing not only the proportion but also giving the actual figures, namely, 0.0093 ampere and 0.0057 ampere.

The denominators of the two fractions are the sum of the resistances of the batteries and of the external resistances in each of the two cases.

Example. Compare the e.m.f.'s producing in two conductors of $\frac{1}{19}$ and 29 ohms resistance respectively currents of 200 and 1.1 amperes.

Solution. By the third proportion we find

$$E_1 : E_2 : : 200 \times \frac{1}{19} : 29 \times 1.1 = 10.52 : 31.9,$$

which as before gives the actual values as well as the ratio of the e.m.f.'s.

Fall of Potential. The e.m.f. of a generator, such as a battery or dynamo, is the potential difference maintainable between its terminals when no current is being taken from it. When a circuit is open the potential difference between its ends is equal to that at the terminals of the battery or generator. When a circuit is closed and active the potential difference between any two points upon it depends upon the current maintained in it.

The potential difference existing between two points of a circuit is often called drop of potential, potential drop, fall of potential, voltage, and the like.

By Ohm's law the potential drop in any part of an active circuit is equal to the resistance of that portion multiplied by the current. Calling the potential drop in a portion of the circuit E', and calling the resistance of the same portion R', we have

$$E' = R'I.$$

The current is the same in all parts of a circuit, hence the current is indicated by the letter I without any mark.

***RI* Drop.** The fall of potential in a portion of an active circuit as just explained and as expressed by this formula is very often called the "RI drop." The RI drop is the fall of potential in any part of an active circuit the resistance of which part is R. As it applies to any part of a circuit it logically includes the drop in the whole circuit.

Example. A circuit is passing a current of 10 amperes. What is the RI drop in a portion of resistance $1\frac{3}{7}$ ohms?

Solution. Applying the formula, we multiply the current by the resistance,

$$R \times I = 10 \times 1\tfrac{3}{7} = \frac{100}{7} = 14.3 \text{ volts}.$$

Example. Calculate the RI drop which will exist in a resistance of 11 ohms added in series to a circuit of 5 ohms and 7 volts.

Solution. When 11 ohms are added to the circuit the resistance of the circuit will be $11 + 5 = 16$ ohms. The current will be $7 \div 16 = 0.4875$ ampere. The *RI* drop of the 11-ohm portion of the circuit will be, by the formula, $11 \times 0.4875 = 5.2625$ volts.

Example. What must the resistance of a 20-volt circuit be if an additional resistance of 5 ohms introduced in circuit has an *RI* drop of 3 volts?

Solution. Applying the formula, we have

$$E = R \times I, \text{ or } 3 = 5 \times I,$$

and therefore the current in the circuit after the extra resistance is introduced is

$$I = \tfrac{3}{5} \text{ ampere.}$$

The same formula will give the resistance of the whole circuit after the additional resistance is introduced. Here E is 20 volts and I is $\frac{3}{5}$ ampere. Substituting gives

$$20 = R \times \tfrac{3}{5} \text{ and } R = 20 \times \tfrac{5}{3} = 33.33 \text{ ohms.}$$

The original resistance of the circuit was therefore $33.33 - 5 = 28.33$ ohms. This is the answer to the problem.

Example. What must the e.m.f. of a circuit be if an additional resistance of 7 ohms develops within itself an *RI* drop of 9 volts, the original resistance of the circuit being 12 ohms?

Solution. With the new resistance the total resistance of the circuit will be $12 + 7 = 19$ ohms. The current in the new part of the circuit can be calculated from the formula as before:

$$E = RI, \text{ or } 9 = 7 \times I, \text{ and } I = \tfrac{9}{7} \text{ amperes.}$$

The e.m.f. of the circuit is given by the same formula:

$$E = RI, \text{ or } E = 19 \times \frac{9}{7} = \frac{171}{7} = 24.43 \text{ volts.}$$

Example. The total resistance of a circuit is 29.37 ohms. The e.m.f. of the generator is 17 volts. Calculate the *RI* drop of the whole circuit.

Solution. By Ohm's law the current is equal to $17 \div 29.37 = 0.58$ amperes. Multiplying the current by the resistance as before we have for the *RI* drop $29.37 \times 0.58 = 17$ volts. The *RI* drop of the circuit is equal to the e.m.f. of the generator, as it should be.

Counter Electro-motive Force. If generators, such as batteries or dynamos, are put in series with each other and are of opposite polarities, which means that they tend to produce currents in opposite directions, the e.m.f. produced by the combination will be equal to the difference of the sum of the e.m.f.'s of the generators working in one direction and the sum of the e.m.f.'s of the generators working in the other direction.

If positive signs are assigned to e.m.f. of one polarity and negative signs to the other, the resulting e.m.f. will be equal to the algebraic sum of the e.m.f.'s. Generators thus placed are in opposition to each other. The e.m.f. of one generator tends to produce a current in the opposite direction to that which the generator in opposition tends to produce. A pair of battery cells connected in opposition would have the zinc plates connected, and the terminals from the other plates connected to the two terminals of the line.

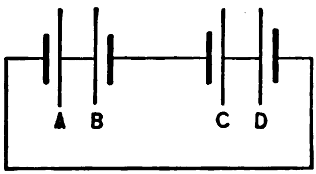
Example. Two cells of battery are connected in series and in opposition. One cell has 1.05 volts e.m.f.; the other has 1.79 volts. What is the net e.m.f.?

Solution. Subtracting the e.m.f. of one battery from that of the other gives net e.m.f. $= 1.79 - 1.05 = 0.74$ volt.

Example. Four cells *A*, *B*, *C*, and *D* are in series with each other. Their e.m.f.'s are respectively 1 volt, $1\frac{1}{2}$ volts, 2 volts, and 1.66 volts. The zinc plate of *A* is connected to the zinc plate of *B*, the carbon plate of *B* to the carbon plate of *C*, and the zinc plate of *C* to the zinc plate of *D*. The circuit is completed by a wire connecting the carbon plate of *D* to the carbon plate of *A*. What is the e.m.f ?

Solution. Cells *A* and *C* are of the same polarity; cells *B* and *D* are of opposite polarity to that of *A* and *C*, but are of the one polarity necessarily with each other. Call the polarity of *B* and *D* positive and that of *A* and *C* negative. The e.m.f. of *E* and *D* then is $1\frac{1}{3} + 1.66 = 3$ volts positive. The e.m.f. of *A* and *C* is $1 + 2 = 3$ volts negative. The net e.m.f. is equal to the difference of the positive and negative e.m.f.'s, namely, $3 - 3 = 0$, or zero.

The conditions of the problem are illustrated in the diagram



Example. Two batteries are placed in opposition to each other. One has a resistance of 2.14 ohms and a voltage of 1.01. The other has a resistance of 5 ohms and a voltage of 2.07. They are connected in series in a circuit of 209 ohms external resistance. Calculate the current.

Solution. The total resistance is $2.14 + 5 + 209 = 216.14$ ohms. As the batteries are in opposition, the e.m.f. of the two is found by subtracting the e.m.f. of one from that of the other. This gives

$$2.07 - 1.01 = 1.06 \text{ volts.}$$

Applying Ohm's law we have

$$I = \frac{1.06}{216.14} = 0.00497 \text{ ampere.}$$

PROBLEMS.

The same e.m.f. acts upon circuits of 10 and of 3 ohms resistance. What is the proportion of the currents resulting?

Ans. $1/10 : 1/3 = 1 : 3.33$, or the current in the 3-ohm circuit will be 3.33 times as strong as in the 10-ohm circuit.

The same e.m.f. can produce currents of 11 and of 5 amperes through different circuits. What are the relative resistances of the circuits?

Ans. 1/11 : 1/5 = 1 : 2.2, or the current in the 5-ohm circuit is 2.2 times as strong as that in the 11-ohm circuit.

What is the relative strength of current produced through identical resistances by 5 and by 7 volts? *Ans.* 5 : 7 = 1 : 1.4.

What will be the relative resistances of circuits in which the above e.m.f.'s will produce currents of identical intensity? *Ans.* 1 : 1.4.

The e.m.f. expended on a portion of an active circuit is 23 volts and the resistance of this portion is $\frac{5}{29}$ ohm. What e.m.f. would be expended on a portion of the circuit of 1½ ohms resistance?

Ans. 200.1 volts.

A current is passing through two incandescent lamps in series. One lamp has a resistance of 107 ohms, the other has a resistance of 200 ohms. The e.m.f. expended on the lamp of higher voltage is found by voltmeter to be 73 volts. What e.m.f. is expended on the other? *Ans.* 39.055 volts.

A circuit has a total resistance of 3 ohms and includes a generator of 2 volts e.m.f. Into this circuit a generator of 1 volt in opposition is introduced whose resistance is 3 ohms. What effects are produced?

Ans. New voltage, 1.
New resistance, 6 ohms.
Original current, ⅔ ampere.
New current, ⅙ ampere.

A generator has 2 ohms resistance and 18 volts e.m.f. It sends a current of 2 amperes through a wire of 4 ohms resistance and also through a battery of 1 volt e.m.f. placed in opposition to the generator. What is the resistance of the battery? *Ans.* 3½ ohms.

A storage battery of 20 volts and 0.33 ohms has been producing a current of 15 amperes through a circuit. Another battery of 20 cells, each cell of 1.08 volts, can produce a current of 2.7 amperes through the same circuit. If the two batteries are connected in opposition what will the data of the circuit be?

Ans. Resistance, 8.33 ohms.
Electro-motive force, 1.6 volts.
Current, 0.192 amperes.

A generator sends 10 amperes through 25 ohms. It can send only 9 amperes through 29 ohms. What is its internal resistance?

Ans. 11 ohms.

This problem can be done by the use of the following equations:

$$\frac{E}{x+25} = 10 \quad (1); \quad \frac{E}{x+29} = 9. \quad (2)$$

There are two storage batteries of identical constants E and r. A single one produces a current of 2 amperes in a circuit of 30 ohms. If the two batteries are connected in parallel the current becomes 2.2 amperes. What are the constants of the batteries?

Ans. 73.3 volts.
6.6 ohms.

The above example can be done by the use of the following equations: $30 + r = \frac{E}{20}$ (1); $30 + \frac{r}{2} = \frac{E}{22}$ (2); in which E and r are the unknown quantities.

A battery has an e.m.f. of 3.21 volts. The potential difference between its terminals within the battery when they are connected by a wire of 8 ohms resistance is 1.07 volts. What is the resistance of the battery?

Ans. 16 ohms.

The above example can be done by algebra by the use of the following equation:

$$\frac{3.21}{8+x} = \frac{1.07}{8}, \text{ in which } x \text{ is the resistance of the battery.}$$

The RI drop in a conductor is 57.25 volts; the current is 0.75 ampere; what is its resistance?

Ans. 76.33 ohms.

CHAPTER V.

RESISTANCE.

Linear Conductors. — Resistance and Weight of Linear Conductors. — Parallel Conductors. — Distribution of Current in Parallel Conductors. — Resistances of Conductors in Parallel. — Combined Resistance of Two Conductors. — Combined Resistance of any Number of Conductors. — Specific Resistance. — Circular Mils. — Effect of Temperature of Conductors on their Resistance. — Problems.

Linear Conductors. A linear conductor is one of uniform cross section. An ordinary wire is an example.

The resistance of linear conductors of identical material varies directly with the length and inversely with the cross-sectional area. Consequently it varies inversely with the square of identical elements of the cross section. Thus in the case of circular-section conductors, such as wires, the resistance varies inversely with the square of the diameter or the square of the radius.

This gives the following proportions, in which l, R, A, d, and r indicate the length, resistance, area, diameter, and radius respectively of conductors, subscript numbers serving to indicate individual conductors.

$$R : R_1 : : \frac{l}{A} : \frac{l_1}{A_1} : : \frac{l}{d^2} : \frac{l_1}{(d_1)^2} : : \frac{l}{r^2} : \frac{l_1}{(r_1)^2}. \qquad (1)$$

$$A : A_1 : : d^2 : (d_1)^2 : : r^2 : (r_1)^2 : : \frac{l}{R} : \frac{l_1}{R_1}. \qquad (2)$$

$$l : l_1 : : RA : R_1A_1 : : Rd^2 : R(d_1)^2 : : Rr^2 : R_1(r_1)^2. \qquad (3)$$

Example. Compare the resistance of two conductors whose lengths are 341 and 361 feet respectively and whose cross-sectional areas are 27 and 37 mils respectively.

Solution. Substituting these values in formula (1) gives

$$R : R' : : \frac{341}{27} : \frac{361}{37} = 126 : 97,$$

which gives the relative resistance of the two conductors. The conductor R' has $\frac{97}{126}$ the resistance of the conductor R.

Example. Compare the resistances of two wires, one of length 3 and diameter 2, the other of length 6 and diameter 8. Call diameter d.

Solution. — Here formula (1) is applicable, substituting for areas of cross sections the squares of like cross-sectional elements. In this case it is the square of the diameter which is to be used. This gives

$$R : R' : : \frac{l}{d^2} : \frac{l}{d_1^2} = \frac{3}{2^2} : \frac{6}{8^2} = \frac{3}{4} : \frac{6}{64} = 8 : 1.$$

The resistance of R is eight times that of R'.

Example. — A wire is to be installed in place of one 0.357 inch in diameter, the new wire to be of one-half the resistance of the original one. Calculate its diameter.

Solution. Applying proportion (2), using the square of the diameter in place of the cross-sectional area, we have

$$A : A' : : d^2 : d_1^2 : : (357)^2 : x^2 : : \frac{1}{1} : \frac{1}{\frac{1}{2}} = 1 : 2,$$

or x^2 must be twice as large as the square of the original wire, which is $(357)^2 = 127,449$. Twice this, or 254,898, is the square of the diameter of the new wire; its diameter is $\sqrt{254,898} =$ 504 mils.

Example. There are two wires of relative resistances 57 and 91 and of relative diameters 31 and 79 respectively. The first wire is 673 feet long. How long is the other?

Solution. — Applying proportion (3), substituting the square of the diameters for the cross-sectional areas, we have

$$673 : x :: 57 \times (31)^2 : 91 \times (79)^2 = 54{,}777 : 567{,}931;$$

therefore by the rule of ratio and proportion

$$x = \frac{673 \times 567{,}931}{54{,}777} = \frac{382{,}217{,}563}{54{,}777} = 6{,}977 \text{ feet.}$$

Example. 1,000 feet of circular wire have a resistance of 105.6 ohms. What will the resistance R of a square wire of the same diameter be?

Solution. From geometry we know that the area of a circle is 0.7854 times the area of the square of the same diameter. Proportion (1) is applicable. As the lengths of the two wires are the same the proportion is simplified by substituting for l in the numerator of the fraction unity, or 1. This gives

$$R : 105.6 :: \frac{1}{1} : \frac{1}{0.7854} \text{ and } R = 105.6 \times 0.7854 = 82.9 \text{ ohms.}$$

Example. A rectangular conductor has a cross section of $\frac{3}{4}$ inch by 2 inches and is 20 feet long. Compare its resistance with that of a round bar 2 inches in diameter and 21 feet long.

Solution. The cross-sectional area of the rectangular bar is $\frac{3}{4} \times 2 = 1\frac{1}{2}$ square inches. The cross-sectional area of the circular bar is 3.14 square inches. Applying proportion (1) and calling the resistance of the rectangular bar 1, we have

$$1 : x :: \frac{20}{1\frac{1}{2}} : \frac{21}{3.14},$$

whence

$$x = \frac{21}{3.14} \div \frac{20}{1\frac{1}{2}} = 0.500,$$

or the rectangular bar has one-half the resistance of the round one.

The proportions on page 53 can be expressed as equations thus:

$$R = \frac{R_1 l A_1}{A l_1} \text{ (1)}; \quad A = \frac{A_1 l R_1}{R l_1} \text{ (2)}; \quad l = \frac{l_1 R A}{R_1 A_1} \text{ (3)}.$$

Example. A conductor of 117 feet in length and 980 square mils in cross-sectional area has a resistance of 1 ohm. What is the resistance of 1,987 feet of wire of 9,862 square mils cross-sectional area?

Solution. — Substituting in formula (1) the values of the problem gives

$$R = \frac{1 \times 1{,}987 \times 980}{9{,}862 \times 117} = 1.69 \text{ ohms.}$$

Example. A conductor 75 feet long and 39 square mils. cross-sectional area is of 20 ohms resistance. What is the cross-sectional area of a conductor of 19 ohms resistance and 89 feet long, of the same alloy?

Solution. Applying formula (2) we have

$$A = \frac{39 \times 89 \times 20}{19 \times 75} = 48.7 \text{ square mils.}$$

Example. A conductor is 1,000 feet long, 150 square mils. in cross-sectional area, and has a resistance of 53 ohms. Another conductor is of 185 square mils cross-sectional area and of 75 ohms resistance. What is its length?

Solution. Applying formula (3) we have

$$l = \frac{1{,}000 \times 75 \times 185}{53 \times 150} = 1{,}745 \text{ feet.}$$

Resistance and Weight of Linear Conductors. The resistance of a wire may be referred to its weight and length. The weights of equal lengths of wire are in proportion to their cross-sectional areas and to the reciprocals of their resistances. If the relative resistances of two wires or other linear conductors and of known length and weight are to be calculated, the weights of equal lengths are found and the calculation is based on them as if they were cross-sectional areas.

$$R : R_1 :: \frac{l}{\text{wt. per foot}} : \frac{l_1}{\text{wt.}_1 \text{ per foot}}.$$

Example. Two bars are 33 and 31 feet long. The first weighs 31 pounds, the second weighs 23 pounds. Calculate their relative resistances.

Solution. One foot of the first bar weighs $31/33 = 0.939$ pounds, and one foot of the second weighs $23/31 = 0.742$ pounds. Dividing the length of each bar by the weight of one foot as determined above gives the relative resistances. $33 \div 0.939 = 35.14$ and $31 \div 0.742 = 41.78$ are the relative resistances. The proportion can then be written

$$35.14 : 41.77 :: 1 : x = 1.19-,$$

or, if we call the resistance of the 33-foot bar unity, the resistance of the 31-foot bar is 1.19–.

Let l be the length of a conductor and w be its weight. Then its weight per unit of length is w/l. Its relative resistance is given as we have seen by dividing the length by the weight per unit of area. This is

$$l \div \frac{w}{l} = l \times \frac{l}{w} = \frac{l^2}{w}.$$

The relative resistance of a conductor is expressed by the quotient of the length squared divided by the weight.

The problem given above can be solved by this expression directly. Substituting the values of the weights and lengths of each conductor in the formula we have for the relative resistances $(33)^2 \div 31 = 35.1$ and $(31)^2 \div 23 = 41.8$, and $351 : 41.8 :: 1 : 1.19$ as before.

Example. There are two wires. 96 feet of one weighs 0.28 pound, 118 feet of the other weighs 0.30 pound. 100 feet of the first have a resistance ot 1 ohm. How many feet of the other will have a resistance of 1 ohm?

Solution. Applying the formula, the relative resistances are

$$\left(\frac{l^2}{w}\right) = 96^2 \div 0.28 = 32{,}914 \text{ and } 118^2 \div 0.30 = 46{,}413.$$

As the lengths of linear conductors are directly proportional to their relative resistances, we have

$$32{,}914 : 46{,}413 :: 100 : x = 141 \text{ feet.}$$

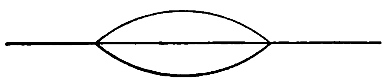
This rule is based upon the weight of equal lengths of the conductors. It applies to any lineal conductors, irrespective of the shape of their cross sections.

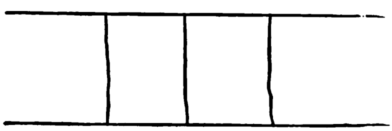
Example. 3 feet of bus bar weigh 5.25 pounds. Calculate the resistance.

Solution. Consulting a wire-resistance table, any wire may be selected as a standard. Thus 0000 wire weighs 639.6 pounds to the 1,000 feet, with a resistance of 0.051 ohm. Therefore 3 feet of this wire weigh $0.6396 \times 3 = 1.9188$ pounds and have a resistance of $0.000{,}051 \times 3 = 0.000{,}153$ ohm. The inverse proportion then holds:

$$5.25 : 1.9188 :: 0.000{,}153 : x = 0.000{,}056 \text{ ohm.}$$

Parallel Conductors. A group of conductors which start in one point and terminate in another are in parallel. This condition is shown in the figure. If a number of conductors are connected across the space between two leads of low resistance compared to that of the conductors, they are also in parallel. This condition is shown in the lower figure.





The correct conception of parallel conductors is that the potential difference maintained between their ends is identical for all.

Distribution of Current in Parallel Conductors. The current passing through each of a set of parallel conductors is calculated by Ohm's law. It is therefore inversely proportional to the relative resistance of the conductor in question referred to that of the others. As conductance is the reciprocal of resistance, if conductance is used in the calculation the proportion becomes a direct one. The general statement may be put thus:

The current passing through single conductors of a set of parallel conductors varies directly with the reciprocal of the individual resistances. If the resistances of the members of a set of parallel conductors are denoted by a, b, c, ... n, the reciprocals of the resistances will be denoted by $1/a$, $1/b$, $1/c$, ... $1/n$. The sum of the reciprocals of all the conductors will be denoted by $1/a + 1/b + 1/c \ldots + 1/n$. The proportion expressing the law just given is the following:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \cdots + \frac{1}{n} : \frac{1}{a} :: 1 : x.$$

Example. Assume that there are three conductors of resistances 2, 3, and 4 ohms respectively which are connected in parallel. If a current is passed through them it will be divided among them. The proportion of current which will pass through each of them is to be determined.

Solution. Taking the reciprocals of the resistances and applying the proportion as above gives

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} : \frac{1}{2} :: 1 : x, \text{ and}$$

$$\frac{12+8+6}{24} : \frac{12}{24} :: 1 : x = \frac{12}{26} \text{ of the whole.}$$

This gives the proportion of the current which will pass through the conductor of conductance, $1/2$ *i.e.* of resistance, 2 ohms. Proceeding in like manner for the other conductors,

we find that the current through the conductor of resistance, 3 ohms, is $\frac{8}{26}$ and that the current through the conductor of resistance, 4 ohms, is $\frac{6}{26}$ of the whole.

By this method is found the relative current which will go through each of a set of parallel conductors. If the total current is known it is only necessary to multiply it by the respective fractions to determine the current in amperes which will pass through each conductor.

Example. In the case just calculated assume that a total current of 19 amperes passes through the conductors. It is then increased to 22 amperes. Calculate the current which will pass through each conductor in each case.

Solution. For the first case proceed as follows:

$$\frac{12}{26} \times 19 = 8.77 \text{ amperes.} \quad \frac{8}{26} \times 19 = 5.85 \text{ amperes.}$$

$$\frac{6}{26} \times 19 = 4.39 \text{ amperes.}$$

For the second case proceed in the same way.

$$\frac{12}{26} \times 22 = 10.15 \text{ amperes.} \quad \frac{8}{26} \times 22 = 6.77 \text{ amperes.}$$

$$\frac{6}{26} \times 22 = 5.08 \text{ amperes.}$$

The wire of 2 ohms resistance will pass of the first current (19 amperes) 8.77 amperes; of the second current (22 amperes) 10.15 amperes. The 3-ohms wire will pass of the same respective currents 5.85 amperes and 6.77 amperes. The third wire will pass of the same 4.39 and 5.08 amperes.

As a test of the correctness of the operations add the currents thus:

$$8.77 + 5.85 + 4.39 = 19 \text{ amperes and}$$

$$10.15 + 6.77 + 5.08 = 22 \text{ amperes.}$$

The fractions of the total current adding up to the total current in each case goes to prove the correctness of the operations.

It is evident that it would have given the same result if the value of the total current in amperes had been used as the third term of each proportion instead of unity. This is the proper procedure when there is question of only one current.

The proportion can also be expressed as a formula or equation. Calling the total current I and the fractional currents $I_a, I_b, I_c, \ldots I_n$, and the resistances of the parallel conductors $a, b, c, \ldots n$, as before, the proportion and the equation become

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\cdots+\frac{1}{n} : \frac{1}{a} :: I \text{ (total current)} : I_a. \qquad (1)$$

I_a (current in conductor a)

$$= \frac{I \text{ (total current)}}{a} \div \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\cdots+\frac{1}{n}\right). \qquad (2)$$

For currents in conductors b, c, and any others substitute the resistance of the conductor whose current is to be calculated for that of a in the second term of the proportion ($1/b$, $1/c$, etc., instead of $1/a$). In equation (2) substitute b, c, or whatever the resistance of the particular wire may be for a in the fraction $1/a$ in the second member of the equation, so that it will read thus:

$$I_b \text{ or } I_c \text{ (or whatever it may be)} = \frac{1}{b \text{ or } c} \cdots n$$
$$\div \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\cdots\frac{1}{n}\right).$$

The same may be expressed in the form of a rule.

The current passing through one of a group of parallel conductors is equal to the total current divided by the resistance of the conductor, which quotient is again divided by the sum of the reciprocals of the resistances of the individual conductors.

Example. A current of 27 amperes passes through three conductors in parallel of the respective resistances of 5, 7, and 9 ohms. Calculate the current which will go through each one.

Solution. Substituting in formula (2) gives

$$I_5 = \frac{27}{5} \div \left(\frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right) = \frac{27}{5} \div \frac{143}{315} = 11.9$$

and consequently for the other two conductors

$$I_7 = \frac{27}{7} \div \frac{143}{315} = 8.5 \; ; \; I_9 = \frac{27}{9} \div \frac{143}{315} = 6.6.$$

Adding the results together as a test of the correctness of the operations we obtain

11.9 + 8.5 + 6.6 = 27, going to prove the correctness of the work.

The wire of 5 ohms resistance passes 11.9 amperes, the wire of 7 ohms resistance passes 8.5 amperes, and the wire of 9 ohms resistance passes 6.6 amperes.

The operation would be shortened by multiplying 27 by $\frac{315}{143}$, which gives 5.95. Then the resistance of each wire divided into this figure gives the current of that wire. Thus:

59.5 ÷ 5 = 11.9; 59.5 ÷ 7 = 8.5; 59.5 ÷ 9 = 6.6, as before.

Example. Four parallel conductors have resistances of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{7}$, and $\frac{1}{9}$ ohm respectively. A current of $\frac{1}{7}$ ampere passes through them. Calculate the current passing through each one.

Solution. Applying formula (2) gives

$$I_{\frac{1}{2}} = \frac{1/7}{1/2} \div (2 + 3 + 7 + 9) = \frac{1/7}{21} \div \frac{1}{2} = \frac{2}{147}.$$

$$I_{\frac{1}{3}} = \frac{1/7}{21} \div \frac{1}{3} = \frac{3}{147}.$$

$$I_{\frac{1}{7}} = \frac{1/7}{21} \div \frac{1}{7} = \frac{7}{147}.$$

$$I_{\frac{1}{9}} = \frac{1/7}{21} \div \frac{1}{9} = \frac{9}{147}.$$

Adding together these currents as a test of the calculation we obtain

$$\frac{2}{147}+\frac{3}{147}+\frac{7}{147}+\frac{9}{147}=\frac{21}{147}=\frac{1}{7} \text{ ampere.}$$

As the sum of the partial currents is equal to the total current the test goes to prove the correctness of the operation.

Example. Assume the resistances of four parallel conductors to be $\frac{2}{3}$, $\frac{3}{7}$, $\frac{4}{5}$, and $\frac{5}{9}$ ohm respectively. Calculate the distribution of a current of $\frac{2}{7}$ ampere among them.

Solution. Proceeding as before we find

$$I_{\frac{2}{3}}=\frac{2/7}{2/3}\div\left(\frac{3}{2}+\frac{7}{3}+\frac{5}{4}+\frac{9}{5}\right)=\frac{2/7}{2/3}\div\frac{826}{120}=\frac{180}{2,891}.$$

In the same way we find

$$I_{\frac{3}{7}}=\frac{280}{2,891};\ I_{\frac{4}{5}}=\frac{150}{2,891};\ I_{\frac{5}{9}}=\frac{216}{2,891}.$$

The sum of these fractions is

$\frac{826}{2,891}=\frac{2}{7}$ ampere, as before, going to prove the correctness of the process.

In this problem the operations reduce to multiplying $\frac{2}{7}\div\frac{826}{120}=\frac{240}{5,782}=\frac{120}{2,891}$ by the reciprocals of the resistances of the conductors. Thus $\frac{120}{2,891}\times\frac{3}{2}=\frac{360}{5,782}=\frac{180}{2,891}$ ampere current, for the conductor of $\frac{2}{3}$ ohm resistance, as before. The same method can be applied to the others.

Resistance of Conductors in Parallel. If several conductors are placed in parallel their combined resistance will be less than that of any single one. The sign of combination of the resistances of parallel conductors is a semicolon placed

between the figures or letters indicating the resistances of the respective conductors.

Thus a ; b indicates the combined resistance of two parallel conductors, one of resistance a, the other of resistance b, when in parallel. The combined resistance is always less than the resistance of either one of the conductors.

The power of conducting electricity is termed conductance. It is proportional to the cross-sectional area of a conductor. If therefore two conductors are in parallel with each other, the conductance of two is equal to the sum of their conductances.

As the conductance of a conductor is equal to the reciprocal of its resistance, the conductance of several conductors in parallel with each other is equal to the sum of the reciprocals of their resistances. It follows that the reciprocal of this sum is equal to the resistance of the conductors in parallel.

To calculate the resistance of conductors in parallel, add together the reciprocals of their resistances. The reciprocal of this sum will be the resistance of the conductors in parallel.

Example. Combine the resistances 2 ; 4 ; 6 ohms.

Solution. The conductances of the conductors in question are $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{6}$ practical units (sometimes but very seldom called mhos). As they are in parallel their conductances must be added to obtain their aggregate conductance.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{24 + 12 + 8}{2 \times 4 \times 6} = \frac{44}{48} = \frac{11}{12}.$$

As resistance is the reciprocal of conductance, the combined resistances 2 ; 4 ; 6 ohms are the reciprocal of this conductance, $\frac{12}{11}$ or 1.0909 ohm.

Example. Calculate the resistance of three conductors in parallel of $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{6}$ ohm resistance respectively.

Solution. The conductances of the conductors are 2, 4, and 6 ohms. Their united conductance is $2 + 4 + 6 = 12$ units. The resistance is the reciprocal of the conductance, or $\frac{1}{12}$ ohm.

Example. What is the resistance of 7, 9, and ½ ohms in parallel?

Solution. $\frac{1}{7} + \frac{1}{9} + 2 = 2\frac{16}{63} = \frac{142}{63}$ = the conductance. The resistance is the reciprocal of the conductance, which is $\frac{63}{142}$ $= 0.444$ ohm.

Combined Resistance of Two Conductors. — Suppose two resistances are to be combined in parallel. Call them a and b. Their reciprocals are $1/a$ and $1/b$. Adding these by the rule for fractions gives $\frac{a+b}{a \times b}$, which is the conductance and whose reciprocal is the resistance. This is $\frac{a \times b}{a+b}$.

To calculate the resistance of two conductors in parallel, divide the product of their resistances by the sum.

Example. Calculate the resistance of 45 and 61 ohms in parallel.

Solution. By the rule just given it is $(45 \times 61) \div (45 + 61)$ $= 2{,}745 \div 106 = 25.9$ ohms.

Example. Calculate the resistance of ½ and ⅔ ohm resistances in parallel.

Solution. As before $\left(\frac{1}{2} \times \frac{2}{3}\right) \div \left(\frac{1}{2} + \frac{2}{3}\right) = \frac{2}{6} \div \frac{7}{6} = \frac{12}{42} = \frac{2}{7}$ $= 0.285$ ohm.

The following may be given as a general arithmetical or algebraic law:

To divide the product of two fractions by their sum, multiply their numerators together for a new numerator; for a new denominator multiply the numerator of one by the denominator of the other, and vice versa, and add the products. This rule gives an easy way of combining the resistances of two conductors in parallel.

Example. Calculate the resistance of ⅓ and 3/7 ohm in parallel.

Solution. Applying the rule we have

$$\frac{4}{5}\ ;\ \frac{3}{7} = \frac{4 \times 3}{(4 \times 7) + (3 \times 5)} = \frac{12}{43} \text{ ohm.}$$

Example. Combine the resistances $\frac{4}{5}$; 3.

Solution. An integral number can be expressed as a fraction having unity for its denominator and the number as its numerator; therefore

$$\frac{4}{5}\ ;\ 3 = \frac{4}{5}\ ;\ \frac{3}{1},$$

and applying the rule gives

$$\frac{4}{5}\ ;\ \frac{3}{1} = \frac{4 \times 3}{(4 \times 1) +\ 3 \times 5)} = \frac{12}{19} \text{ ohm.}$$

Combined Resistance of any Number of Conductors. The rule for the calculation of the resistance of any number of resistances in parallel may be thus deduced. Assume resistances a, b, c, and d. Their reciprocals are $1/a$, $1/b$, $1/c$, and $1/d$. These are their conductances. Adding them gives the following expression:

$$\frac{abc + abd + acd + bcd}{abcd},$$

whose reciprocal is the combined resistance, which is

$$\frac{abcd}{abc + abd + acd + bcd}.$$

Expressed in words this may be thus put:

To calculate the resistances of any number of resistances in parallel, multiply the resistances together for a numerator. For a denominator multiply together all possible combinations of one less than the given resistances and add the products. There will be as many such products as there are resistances. The result is the combined resistance.

Example. Calculate the resistances of the following resistances in parallel: $\frac{1}{2}$, 2, 3, and $\frac{2}{3}$ ohm.

Solution. By the rule given above we obtain the expression

$$\frac{1/2 \times 2 \times 3 \times 2/3}{(1/2 \times 2 \times 3)+(1/2 \times 2 \times 2/3)+(1/2 \times 3 \times 2/3)+(2 \times 3 \times 2/3)}$$

$$= \frac{3}{13} \text{ ohm.}$$

The operation of combining these resistances may be thus expressed:

$$a\,;\,b\,;\,c\,;\,d = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}.$$

It is obvious that when there are many conductors in parallel the calculation of their combined resistance by the general rule would be complicated and open to error in calculation. The following is a good method to use in doing the calculation, or as a check on the operation if done by the other method.

Combine two resistances by the regular rule. Then combine the result with another of the resistances, and combine this result with yet another of the resistances, and so on until the last one has been combined.

Example. Apply this method to the problem given above.

Solution. The resistances are $\frac{1}{2}; 2; 3; \frac{2}{3}$. Applying the rule we have

$$\frac{1}{2}; 2 = \frac{1/2 \times 2}{1/2 + 2} = \frac{2}{5}. \quad \frac{2}{5}; 3 = \frac{2/5 \times 3}{2/5 + 3} = \frac{6}{17}. \quad \frac{6}{17}; \frac{2}{3} = \frac{6/17 \times 2/3}{6/17 + 2/3}$$

$= \frac{3}{13}$ ohm. This is the same result as that reached by the other method, and therefore operates as a check upon it. In many cases it is the better method of the two.

Example. Combine the resistances $3; 5; 5; \frac{1}{4}; 9; \frac{17}{18}$ ohms in parallel.

Solution. (*a*) $3;5 = \frac{3 \times 5}{3+5} = \frac{15}{8}$. (*b*) $\frac{15}{8}; 5 = \frac{15/8 \times 5}{15/8 + 5}$ $= \frac{75}{55}$. (*c*) $\frac{75}{55}; \frac{1}{4} = \frac{75/55 \times 1/4}{75/55 + 1/4} = \frac{75/220}{3{,}000/220 + 55/220} = \frac{75}{355}$. (*d*) $\frac{75}{355}; 9 = \frac{75/355 \times 9}{75/355 + 9} = \frac{675}{3{,}270}$. (*e*) $\frac{675}{3{,}270}; \frac{17}{18} =$ $\frac{675/3{,}270 \times 17/18}{675/3{,}270 + 17/18} = \frac{11{,}475}{67{,}740} = \frac{765}{4{,}516}$ ohm.

Specific Resistance. — Substances vary greatly in their power of conducting electricity. The resistance of a prism or cylinder varies directly with its length and inversely with its cross-sectional area. The resistance of a cylinder or prism is therefore equal to

$$(\text{sp. r.}) \frac{\text{length}}{\text{section}}. \qquad (1)$$

In the above expression sp. r. is a coefficient whose value varies with the material of the conductor. It is called the specific resistance or resistivity of the material.

The reciprocal of the above coefficient, 1/sp.r., is the conductivity of the material.

Specific resistance is the resistance between two faces of a cube of the material. In the C.G.S. system the unit cube is one centimeter cube, so that the C.G.S. unit of specific resistance is the resistance of a centimeter cube of the substance.

Call ρ the specific resistance of any substance, and let R denote resistance. The dimensions of resistance are LT^{-1} (see page 122). The resistance of any lineal conductor is equal to

$$\frac{l}{A} \times \rho = \text{resistance} = LT^{-1}.$$

But for l can be substituted L, and for A, L^2, which are the dimensions of length and of area respectively, giving

$$\frac{L}{L^2} \times \rho = LT^{-1}, \text{ or } \quad \rho = L^2T^{-1},$$

which are the dimensions of specific resistance.

In tables of specific resistance it is usually expressed in microhms for solids and in ohms for solutions. Often it is expressed in C.G.S. units of the electro-magnetic or electrostatic system.

Example. Assume the specific resistance of tin to be 13.36 microhms. What is the resistance of a tin wire 350 feet long and $\frac{1}{8}$ inch diameter?

Solution. The cross section of the wire is $3.1416 \times (1/16)^2 = 0.0123$ square inches. $0.0123 \times 6.45 = 0.0793$ square centimeter. The length of the wire is $350 \times 30.48 = 10{,}668$ centimeters. Substituting in expression (1) the values as above gives

$$\text{Resistance} = 13.36 \times \frac{10{,}668}{0.0793} = 1{,}797{,}282 \text{ microhms} = 1.797$$

ohms.

If the diameter, d, of a wire is given instead of the cross-sectional area, the expression for the area of the cross section will be $\pi d^2/4$, because $d^2/4$ is the square of $d/2$. As the resistance of a conductor varies inversely with its cross-sectional area, it will vary with $4/\pi d^2$, which reduces to $1.2732/d^2$. Multiplying this by the length and by the specific resistance gives the expression

$$\frac{\text{sp. res.} \times 1.2732 \times l}{d^2} \qquad (2)$$

for the resistance of a wire of length l and of diameter d.

Example. What is the resistance of a copper wire $1\frac{1}{2}$ meters long, 1 millimeter thick, the specific resistance of copper being taken as 1.652 microhms?

Solution. Substituting in the formula for l and d their values in centimeters and for sp. res. its value 1.652 we have

$$\text{Resistance} = \frac{1.652 \times 1.2732 \times 150}{0.01} = 31{,}550 \text{ microhms, or}$$

0.03,155 ohm.

In calculating the resistance between electrodes in a solution

the facing areas are the only ones usually taken into account, because the rear or remote faces are of comparatively little value in the conduction. The conductor is treated as a prism or, where the electrodes are of unequal size, as a frustum of a pyramid the area of whose bases is determined by the faces of the electrodes.

Example. An electroplater's bath has a specific resistance of 42 ohms. The electrodes are each 140 square inches in area and are 3 inches apart. Calculate the resistance.

Solution. The area of the electrodes is $140 \times 6.45 =$ 903 square centimeters. The distance separating them is $3 \times 2.54 = 7.62$ centimeters. Substituting these values in expression (1) we have

$$R = 42 \times \frac{7.62}{903} = 0.354 \text{ ohm.}$$

Example. In a battery the facing areas of the plates are 12 and 40 square inches respectively and they are $\frac{3}{4}$ inch apart. The specific resistance of the solution is 9 ohms. Calculate the resistance.

Solution. The average area of the plates is $\frac{12 + 40}{2} =$ 26 square inches = 167.7 square centimeters. $\frac{3}{4}$ inch = 1.905 centimeters. Substituting in the formula as before gives

$$R = 9 \times \frac{1.905}{167.7} = 0.102 \text{ ohm.}$$

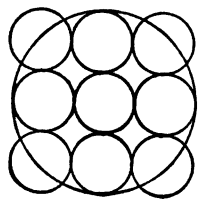
Calculations of the resistance of solutions are not of much accuracy. They may be useful as approximations. The least change in the composition of a solution affects its specific resistance. The approximate formulas given above suffice for most cases.

Circular Mils. A mil is a measure of length; it is one one-thousandth of an inch.

A circular mil is a measure of area; it is the area of a circle one mil in diameter.

The area of a circle may be expressed in circular mils. The area of a circle in circular mils is equal to the number of circular mils expressed or obtained by squaring its diameter given in linear mils. Thus the area of a circle 3 mils in diameter is equal to $(3)^2$ circular mils, or 9 circular mils.

The diagram shows a circle supposed to be of 3 mils diameter. By geometry we know that the area of the circle is 0.7584 that of the circumscribing square. Assume the square to be divided into 9 small squares, each one supposed to be 1 mil in diameter, and whose united areas, or the sum of whose areas, is equal to the area of the large square. These squares are equal in number to the square of the diameter of the circle in circular mils. Within each small square is inscribed a circle whose diameter, under the conditions of the proposition, is



1 mil. The ratio of the area of each small circle to its circumscribing square being the same as that of the large circle to its circumscribing square, and the sum of the areas of the small squares being equal to the area of the large square, it follows that the area of the large circle is equal to the sum of the areas of the small circles. But the number of these is equal to the square of the diameter of the large circle in mils. The area of each small circle is equal to 1 circular mil. Hence it follows that the area of the large circle is equal to a number of circular mils, equal to the diameter in mils squared. What is true for a circle of one size is true for any other, so that the proposition is proved.

The cross-sectional area of wires and circular conductors is often expressed in circular mils. In many cases it is the

most convenient way of working and is in constant use by engineers.

A copper wire 1 mil in diameter and 1 foot long has a resistance at the temperature of 75° F. (24° C.) of 10.79 ohms. This figure is necessarily approximate, because the resistance of different samples of copper varies, but it is in constant use in practical work.

What has been explained gives us the rule:

The resistance of a copper wire is equal to the product of its length in feet by 10.79 divided by the square of its diameter in mils.

$$R = \frac{l \times 10.79}{d^2 \text{ (circ. mils)}}.$$

The square of the diameter in mils of a circle is its actual area in circular mils.

Example. What is the area of a circle ¾ inch in diameter, in circular mils?

Solution. ¾ inch = 1,000 × ¾ mils = 750 mils. This is the diameter of the circle in linear mils. Its area in circular mils is equal to $(750)^2 = 562{,}500$ circular mils.

Example. Calculate the resistance of a copper wire 1,034 feet long and 320 circular mils in area. In such statements as this, by "area" cross-sectional area is always understood.

Solution. Substituting the given values in the formula we have

$$R = \frac{1{,}034 \times 10.79}{320} = 34.865 \text{ ohms.}$$

This can be done by logarithms.

log. 1,034	3.01452
log. 10.79	1.03302
colog. 320	7.49485−10
log. 34.865	1.54239

Example. Calculate the resistance of a copper wire 14 mils diameter and 1,644 feet long.

Solution. $(14)^2 = 196$, the area of the wire in circular mils. Substituting as before gives

$$R = \frac{1,644 \times 10.79}{196} = 90.5 \text{ ohms.}$$

By logarithms:

log. 1,644	3.21590
log. 10.79	1.033302
colog. 196	7.70774−10
log. 90.5	1.95666

Effect of Temperature of Conductors on their Resistance. The resistance of conductors varies with the temperature, increasing as the temperature rises. Tables of the increase are published; the rate of increase varies with the temperature, the rate decreasing as the conductor's temperature rises. If the resistance of a copper conductor at 0° C. be taken as 1, resistance at 20° C. will be 1.07968; at 50° C., 1.20625; at 80° C., 1.33681. These figures give as percentage rates for 1° change in temperature between 0° C. and 20° C. 0.3984; between 20° C. and 50° C. 0.3856; between 50° C. and 80° C. 0.3607 if the resistances at 0°, 20°, and 50° be taken as 100% for each case. These figures are the results of experiments on a certain sample of copper. Other samples would give different results. 0.40 per cent is a convenient approximate coefficient.

Example. If the resistance of a copper wire at 10° C. is 29 ohms, what will its resistance at 19° C. be?

Solution. The change in temperature is $19° - 10° = 9°$. At the temperature of the problem the percentage may be taken as 0.3984, and multiplying this by the number of degrees

gives the total change as a percentage, or $9 \times 0.3984 = 3.586$. This percentage has to be added to the original resistance, 29 ohms. Thus $29 \times 1.03586 = 30.04$ ohms.

Example. What heat is indicated by a copper wire whose resistance at 15° C. is 300 ohms when its resistance rises to 495 ohms as it is placed in a chimney? Take 0.38 as the percentage of change.

Solution. It rises $\frac{195}{300} = 65$ per cent in resistance. Dividing the per cent increase in resistance by the per cent increase for 1° C. gives the number of degrees C. it has risen. Thus $65 \div 0.38 = 171°$ C., to be added to the original temperature; $15° + 171° = 186°$ C.

Example. A mile of copper wire in the open air varies from 7.9 ohms at night to 8.6 ohms in the daytime; what is the difference in temperature?

Solution. The increase in resistance is $\frac{7}{79} = 8.86$ per cent. Taking 0.39 as the per cent per degree, and dividing as before, we have $8.86 \div 0.39 = 22.7°$ C., the increase in temperature.

The following is Matthiessen's formula:

Resistance at temp. $t°$ C. = resistance at 0° C. $\times (1 + at + bt^2)$.

The following are values of a and b:

	a	b
Pure copper,	+0.003,84	+0.000,001,26
Mercury,	0.000,748,5	−0.000,000,398
German silver,	0.000,443,3	+0.000,000,152

PROBLEMS.

Calculate the resistance of two conductors in parallel, one of 5 ohms and the other of 9 ohms resistance. *Ans.* $3\frac{3}{14}$ ohms.

What is the combined resistance of two conductors of $\frac{1}{6}$ and $\frac{1}{7}$ ohm in parallel? *Ans.* $\frac{1}{13}$ ohm.

Combine the resistances $\frac{3}{4}$ ohm and $\frac{7}{8}$ ohm in parallel.

Ans. $\frac{21}{52}$ ohm.

Calculate the resistance of $\frac{3}{7}$ and 9 ohms in parallel.

Ans. $\frac{9}{22}$ ohm.

Combine the following resistances in parallel, $\frac{3}{4}$, $\frac{7}{8}$, and $\frac{9}{10}$ ohm.

Ans. $\frac{189}{678}$ ohm.

Calculate the resistance of 4 conductors in parallel of 1, 3, 7, and 12 ohms resistance respectively. *Ans.* $\frac{84}{131}$ ohm.

What is the resistance of three wires in parallel of conductances of 5, 16, and $\frac{3}{8}$ ohms respectively? *Ans.* $\frac{8}{171}$ ohm.

A resistance coil has a resistance of 1,015 ohms. A resistance is to be placed in parallel with it so as to reduce the combined resistance to 1,000 ohms. What must the second resistance be?

Ans. It is the reciprocal of the difference of the reciprocals, 67,666 ohms.

What is the resistance of a wire of $\frac{1}{5}$ ohm in series with three wires in parallel of $\frac{1}{6}$, $\frac{1}{17}$, and $\frac{1}{8}$ ohm respectively, and with another conductor in series of 16 ohms resistance?

Ans. $16\frac{36}{155}$ ohms.

The resistance of 1,000 feet of wire 0.01 inch in diameter is 105.641 ohms. What is the resistance of 1,000 feet of wire 0.008 inch in diameter? *Ans.* 1.65 ohms.

What is the resistance of 633 feet of wire 0.08 inch in diameter if the resistance of 1,000 feet of wire 0.018 inch in diameter is 33.135 ohms? *Ans.* 1.061 ohms.

1,000 feet of No. 15 wire weigh 9.84 pounds, 1,000 feet of No. 10 wire weigh 31.38 pounds, with a resistance of 1.023 ohms. What is the resistance of the 1,000 feet of No. 15 wire? *Ans.* 3.26 ohms.

Compare the resistance of two conductors of 6,000 and 9,000 circular mils area. *Ans.* 1 to $\frac{2}{3}$.

Assume three conductors of 5, 6, and 8 ohms resistance respectively. A current of 1 ampere flows through them. Calculate its distribution. *Ans.* $\frac{48}{118}$ ampere in 5 ohm lead.
$\frac{40}{118}$ ampere in 6 ohm lead.
$\frac{30}{118}$ ampere in 8 ohm lead.

Calculate the distribution of one ampere of current in 4 conductors in parallel of $\frac{1}{2}$, $\frac{1}{6}$, 7, and 9 ohms resistance respectively.

Ans. $\frac{126}{520}$ ampere in $\frac{1}{2}$ ohm lead.
$\frac{378}{520}$ ampere in $\frac{1}{6}$ ohm lead.
$\frac{9}{520}$ ampere in 7 ohm lead.
$\frac{7}{520}$ ampere in 9 ohm lead.

1,500 feet of conductor of 6,530 circular mils section have a resistance of 2.439 ohms. What is the resistance of 1,700 feet of 5,178 circular mils conductor? *Ans.* 3.48 ohms.

Two wires of identical cross section are 19,173 and 29,871 feet long respectively. The longer wire is of 13.1 ohms resistance. Calculate the resistance of the other. *Ans.* 8.41 ohms.

Compare the resistances of equal lengths of wire of 1,021 and 8,234 circular mils.

Ans. The small one has 8.06 times the resistance of the large one.

Calculate the resistance of a bar of copper, sp. r. 1.652; 15 feet long and 2 square inches cross section.

Ans. 58.53 microhms.
$5{,}853 \times 10^{-8}$ ohms.

Calculate the resistance of a copper wire 114 feet long and 0.010 inch in diameter, taking sp. r. as 1.652.

Ans. 11,332,000 microhms.
11.332 ohms.

A copper wire has a resistance at 16° C. of 257 ohms. Calculate the resistance at 27° C., taking the coefficient as 0.40 per cent.

Ans. 268.3 ohms.

The resistance of a copper conductor changes from 30 ohms to 27.9 ohms at an ordinary temperature. What is the fall in temperature? *Ans.* 17.5° C.

If the resistance of a copper conductor at 50° F. is 295 ohms, what is its resistance at 60° F.? (50° F. = 10° C.; 60° F. = 15.5° C.)

Ans. 301 ohms.

Calculate the relative resistance of German silver at 11° C. and at 15° C., and determine the change in resistance per degree C. Take the resistance at 0° C. as unity.

Ans. At 11° C. 1.004,89; at 15° C. 1.006,68. Increase for 4 degrees 0.001,79; for 1 degree 0.000,45.

What is the resistance at 19° C. of a copper wire whose resistance at 0° C. is 17 ohms?

Applying Matthiessen's formula we have

Resistance at 19° C. $= 17 \text{ ohms} \times [1 + (0.003{,}84 \times 19) + (0.000{,}001{,}26 \times 19^2)] = 18.24$ ohms.

CHAPTER VI.

KIRCHHOFF'S LAWS.

Kirchhoff's Laws. Kirchhoff's laws apply to the distribution of current and e.m.f. in branching conductors and network of conductors when they are passing steady currents. The laws are two in number.

1. When conductors forming parts of an active circuit meet in one point the algebraic sum of the currents is zero, negative signs being assigned to currents leaving the junction and positive signs to those going to the junction, or vice versa. Or put into simpler language, the currents going to a junction are equal in the sum of their intensities to the sum of the intensities of the currents leaving it.

2. When conductors form a circuit comprising within it a source of e.m.f., the sum of the products of the intensity of the current within each part of the circuit into the resistance of the same part is equal to the e.m.f. of the system provided the elements of current be taken in cyclical order.

In other words, Kirchhoff's second law applies when the direction of the currents can be deduced. Otherwise, although it is true, it cannot be used in practice.

Kirchhoff's laws in calculations are used conjointly with Ohm's law. Some simple examples of their use are given here.

Example. Three conductors meet in a point. One conductor carries a current of 3 amperes to the junction, another carries a current of 6 amperes away from the junction. What current does the third conductor carry and in what direction?

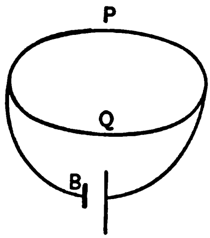
Solution. This case comes under the first law. As the cur-

rents going to the junction must equal those leaving it, the third conductor carries a current of 3 amperes to the junction; thus the currents to the junction are $3 + 3 = 6$ amperes and the current from the junction is thus equal to them.

To do it by algebraic addition, call the current to the junction $+ 3$. Call the current from the junction $- 6$. Let x be the other current. Then by the law we have

$$x + 3 - 6 = 0 \text{ and } x = 3.$$

As the sign of x is positive the current is to the junction.



Example. Calculate the following circuit by Kirchhoff's laws. In the diagram the capital letters indicate the currents in each branch. The diagram may be taken as explaining the circuit and dispensing with the necessity of an explanation.

Solution. Let b, p, q indicate the resistances of the respective branches, b including the resistance of the battery.

From the first law is obtained the expression

$$P + Q = B. \tag{1}$$

The e.m.f. of a circuit is equal to the product of the current by the resistance. (Ohm's law.) In the circuit under consideration the portions B and Q constitute one cycle and the portions B and P another, as it is evident that a consecutive circuit is afforded by each of the two divisions specified. This is what is meant by "taking the elements of current in cyclical order." Applying the second law we obtain

$$Bb + Pp = \text{e.m.f. of the generator.} \tag{2}$$

$$Bb + Qq = \text{e.m.f. of the generator.} \tag{3}$$

In general the sum of the R.I. drops of either of the cycles is equal to the e.m.f. of the system.

This follows from Ohm's and Kirchhoff's laws, as each cycle includes the battery and a full lead from terminal to terminal. The potential difference between the ends of the lead P is the same as that between the ends of the parallel lead Q.

Let the resistances be as follows: $b = 23$; $p = 7$; $q = 11$. Let the e.m.f. be 1. Substituting these values in equations (2) and (3) and writing them out as simultaneous equations with equation (1) as one of the group we have

$$P + Q = B. \quad (4)$$
$$23B + 7P = 1. \quad (5)$$
$$23B + 11Q = 1. \quad (6)$$

Solving these by the ordinary method for simultaneous equations we find

$$P = 0.0224;\ Q = 0.0142;\ \text{and}\ B = 0.0366.$$

To do it by determinants write the equations as follows:

$$23B + 0P + 11Q = 1.$$
$$23B + 7P + 0Q = 1.$$
$$-B + P + Q = 0.$$

Arranging the determinants for the value of B gives

$$\begin{vmatrix} 1 & 0 & 11 \\ 1 & 7 & 0 \\ 0 & 1 & 1 \end{vmatrix} \div \begin{vmatrix} 23 & 0 & 11 \\ 23 & 7 & 0 \\ -1 & 1 & 1 \end{vmatrix} = \frac{18}{491} = 0.0366 \text{ ampere} = B.$$

The determinant dividends for P and Q respectively are

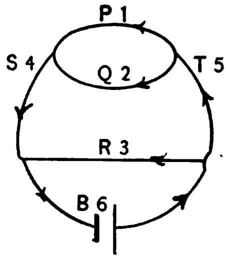
$$\begin{vmatrix} 23 & 1 & 11 \\ 23 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 11 \text{ and } \begin{vmatrix} 23 & 0 & 1 \\ 23 & 7 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 7$$

Giving

$$P = \frac{11}{491} = 0.0224 \text{ ampere, and } Q = \frac{7}{491} = 0.01426 \text{ ampere.}$$

Example. Calculate the circuit of the cuts by Kirchhoff's laws. The capital letters indicate currents, the figures indicate the resistances. Let e.m.f. = 1.

Solution. The first law gives the following equations:



$$B - R - S = 0. \qquad (1)$$

$$S - P - Q = 0. \qquad (2)$$

$$T - P - Q = 0. \qquad (3)$$

From the second law are found the following:

$$6B + 3R = 1. \qquad (4)$$

$$3R - 4S - 2Q - 5T = 0. \qquad (5)$$

$$2Q - P = 0. \qquad (6)$$

Subtracting (2) from (3),

$$T - S = 0. \qquad (7)$$

The equations (1), (2), (4), (5), (6), and (7) are solved as simultaneous equations, giving

$$P = \frac{6}{315}, \qquad R = \frac{29}{315}, \qquad T = \frac{9}{315},$$

$$Q = \frac{3}{315}, \qquad S = \frac{9}{315}, \qquad B = \frac{38}{315}.$$

PROBLEMS.

Three conductors meet in a point or junction. One carries a current of 21 amperes to the junction, another carries a current of 5 amperes to it. What is the current in the other conductor?

Ans. A current of 26 amperes flowing from the junction.

Five conductors meet in a point. Three carry currents of 6, 8, and 9 amperes respectively to the junction; one of the other conductors carries a current of 39 amperes away from the junction. What is the current in the remaining conductor?

Ans. A current of 16 amperes to the junction.

Three conductors *a*, *b*, and *c* meet in a point; *a* carries a current of 7 amperes away from the junction, *b* carries a current of 9 amperes and *c* one of 2 amperes. What is the direction of the currents in *b* and *c*? *Ans.* Current in *b* to the junction; current in *c* away from it.

In the last case the current in *b* rises to 11 amperes and in *c* falls to 1 ampere. What is the current in *a*?

Ans. 10 amperes from the junction.

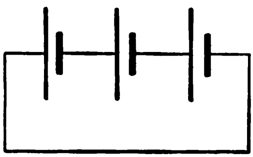
In the same case *c* rises to 5 amperes, *a* remaining unchanged. What happens to *b*? *Ans.* It becomes 15 amperes to the junction.

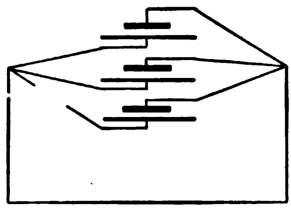
CHAPTER VII.

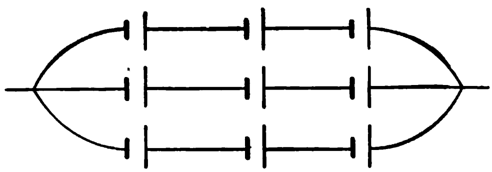
ARRANGEMENT OF BATTERIES.

Electro-motive Force of a Battery. — Resistance of a Battery. — Potential Drop of a Battery. — Greatest Current from a Battery. — Rules for Calculating a Battery. — Energy Expended in a Battery. — Rule for Calculating a Battery of Given Efficiency. — Discussion. — Problems.

Electro-motive Force of a Battery. The electro-motive force of a battery is equal to the sum of the electro-motive forces of its cells in series, irrespective of the number in parallel. Thus







a battery arranged 10 in series and 5 or 6 or any other number in parallel has the e.m.f. of the sum of the e.m.f.'s of the 10 cells. The first of these cuts represents a battery of 3 cells connected in series. The next represents a battery also of 3 cells connected in parallel. The third represents a battery of 9 cells, arranged 3 in series and 3 in parallel.

Resistance of a Battery. The resistance of a battery arranged like a rectangle is equal to the sum of the resistances of the cells in series divided by the number of cells in parallel.

Example. Suppose a battery to consist of 50 cells arranged

5 in series and 10 in parallel. What are its e.m.f. and resistance? Each cell has an e.m.f. of 1.1 volt and a resistance of 4.2 ohms.

Solution. The e.m.f. of the battery is equal to the e.m.f. of a single cell multiplied by the number in series; $1.1 \times 5 = 5.5$ volts. The resistance of the battery is equal to the resistance of a single cell multiplied by the number in series and divided by the number in parallel; $4.2 \times 5 \div 10 = 2.1$ ohms.

Potential Drop of a Battery. The potential drop sometimes called the *RI* drop of a battery, or lost volts, is the difference between the e.m.f. on open circuit and the potential drop between its terminals when connected by a conductor of given resistance. The *RI* drop of the battery varies according to the resistance of the outer circuit or conductor connecting its terminals. It is the e.m.f. expended on maintaining the current through the resistance of the battery.

Example. A voltmeter connecting the terminals of a battery on open circuit reads 24 volts. When the terminals are connected by a conductor the voltmeter shows 20 volts. What is the *RI* drop of the battery when its terminals are connected by the conductor in question?

Solution. It is $24 - 20 = 4$ volts.

The drop of potential of a battery is equal to its e.m.f. multiplied by its resistance and divided by the resistance of the entire circuit.

Example. A battery of 3 volts and 5 ohms is connected in circuit with a resistance of 21 ohms. What is its drop of potential?

Solution. By the rule just given it is equal to $3 \times 5 \div (5 + 21) = 0.577$ volt.

Greatest Current from a Battery. If a given number of cells or equivalent generators are to be used to produce a current through a given resistance, they will produce the greatest current when the external and internal resistances are the same.

Assume 40 cells each of $\frac{1}{2}$ ohm resistance and 1 volt e.m.f.

and assume an external resistance of 5 ohms. If the cells are in series their e.m.f. will be 40 volts and their resistance will be $40 \times 1/2 = 20$ ohms. The current in the circuit will be $\frac{40}{20 + 5} = 1.6$ amperes.

If the cells are 20 in series and 2 in parallel, their e.m.f. will be 20 volts and their resistance 5 ohms, which is that of the outer circuit. The current will be $\frac{20}{5 + 5} = 2$ amperes.

If we go a step further and assume the cells to be 10 in series and 4 in parallel, the internal resistance is 1.25 ohms and the e.m.f. is 10 volts. The current will be $\frac{10}{1.25 + 5} = 1.6$ amperes.

These three cases illustrate the law enunciated above. The greatest current is produced when the battery is so connected as to have its internal resistance equal to that of the outer circuit.

Rules for Calculating a Battery. A general method for calculating the number and arrangement of cells of battery to maintain the greatest current through a given resistance can be based upon the above principle.

Arrange the cells so as to give twice the e.m.f. required and so as to have a resistance equal to that of the external circuit. To effect this it will often be necessary to arrange the cells irregularly. The more regularly the cells are arranged the better will the results be, as a rule.

Example. The constants of a cell of a battery are 1 volt and 4 ohms. The battery is to maintain a current of 0.56 ampere through a circuit of 30 ohms resistance. Calculate the number and arrangement of the cells.

Solution. The e.m.f. required for the outer circuit will be, by Ohm's law, $0.56 \times 30 = 17$ volts. As the internal and external resistance are to be equal to each other, the battery, by the law of the distribution of energy, must maintain double this e.m.f., 17 volts to be expended within itself and 17 volts

to be expended upon the outer circuit. Therefore 34 cells in series will be needed. A single series of 34 cells would have a resistance of $34 \times 4 = 136$ ohms. Four series in parallel would have one-fourth this resistance, or 34 ohms. This would be a close enough approximation for ordinary purposes. Five series in parallel would have a resistance of $136 \div 5 = 27.2$ ohms, again a close enough approximation for most purposes. If a group of 20 in series and 5 in parallel is placed in series with a group of 14 in series and 4 in parallel, the resistance of the battery will be the sum of the resistances of the two groups. The resistance of the first group will be $(20 \times 4) \div 5 = 16$ ohms. That of the second group will be $(14 \times 4) \div 4 = 14$ ohms. The total resistance of the battery will be $16 + 14 = 30$ ohms. The number of cells required is $(5 \times 20) + (4 \times 14) = 156$.

A good general rule is the following:

To calculate the number of cells to produce a given current through a given resistance, group enough cells in parallel to give on short circuit twice the required current. Treating this as a single cell place enough of these groups in series to give twice the potential drop of the outer circuit.

Example. A current of 5 amperes is to be maintained through a resistance of 6 ohms. The battery constants are 1 volt and 4 ohms. Calculate the cells.

Solution. Twice the required current is 10 amperes. As the resistance of a single cell is 4 ohms, 40 cells in parallel will have a resistance of $\frac{1}{10}$ ohm. The voltage of a single cell is 1; the group of 40 cells in parallel will give on short circuit a current of $1 \div \dfrac{1}{10} = 10$ amperes. The drop of the outer circuit is $5 \times 6 = 30$ volts. Twice this is 60, so that 60 groups are needed in series to give the e.m.f. The total number of cells is $60 \times 40 = 2{,}400$.

Testing the correctness by Ohm's law, we have the external

resistance 6 ohms, the internal resistance 6 ohms, the e.m.f. 60 volts, and current $= \frac{60}{6 + 6} = 5$ amperes, showing the correctness of the work.

Energy Expended in a Battery. The energy expended in any part of a circuit is porportional to its resistance. The energy expended in a battery forming part of a circuit and maintaining a current in it is proportional to its resistance also. This energy is wasted.

The proportion of energy wasted in a battery supplying a circuit is equal to the resistance of the battery divided by the total resistance of the circuit.

Example. A battery of 13 ohms resistance supplies a circuit of 19 ohms external resistance. Calculate the proportion of energy wasted.

Solution. The total resistance of the circuit is $13 + 19 = 32$ ohms. This is the divisor of the fraction whose numerator is 13. The energy wasted in the battery is $\frac{13}{32} = 40.6$ per cent.

Rule for Calculating Battery of given Efficiency. To calculate the battery to supply an external circuit of given resistance with a given current at a given percentage of loss, proceed as follows. As the drop in potential is equal to the product of the resistance of the part of the circuit in question multiplied by the current, the energy expended in a battery is proportional to its drop in potential, the same as it is proportional to its relative resistance. Subtract the per cent of loss of energy in the battery from 100. Take the difference as the denominator of a fraction whose numerator is the per cent of loss. The resistance of the external circuit multiplied by this fraction is the resistance of the battery. The potential drop of the external circuit multiplied by the same fraction is the potential drop of the battery. The e.m.f. of the battery

is the sum of the potential drops. Or multiply the total resistance of the circuit by the current. The product will be the e.m.f. of the battery.

Example. Calculate the resistance and e.m.f. of a battery to maintain a current of 3 amperes through a resistance of 18 ohms with an expenditure of 19 per cent of energy in the battery.

Solution. The resistance of the battery is given by the product of the resistance of the outer circuit by the fraction $\frac{19}{100 - 19} = \frac{19}{81}$. $18 \times \frac{19}{81} = 4.22$ ohms. This is the resistance of the battery. The total resistance of the circuit is $18 + 4.22 = 22.22$. The e.m.f. of the battery is equal to the product of the total resistance by the current, 3 amperes, which is $22.22 \times 3 = 66.66$ volts. The last result can be reached by the other method. The potential drop of the outer circuit is equal to the product of the current by the resistance. It is therefore $18 \times 3 = 54$ volts. $54 \times \frac{19}{81} = 12.66$, the potential drop of the battery. The e.m.f. of the battery is $12.66 + 54 = 66.66$ volts, as before.

The following is a somewhat shorter rule. Subtract the per cent of loss from 100 per cent, which gives the per cent of efficiency. Divide the potential drop of the external circuit by the efficiency expressed as a decimal; the quotient is the e.m.f. of the battery. Subtract the potential drop of the external circuit from that of the e.m.f. of the battery; the remainder is the potential drop of the battery.

Example. A current of 29 amperes is to be maintained through 11 ohms resistance with 29 per cent loss. Calculate the e.m.f. of the battery.

Solution. $100 - 29 = 71$, the efficiency. $29 \times 11 = 319$, the potential drop of the outer circuit. $319 \div 0.71 = 450$, the e.m.f. of the battery. $450 - 319 = 131$, the potential drop of the battery.

Discussion of Principles of Calculating Batteries. The whole matter of calculating a battery to give the exact resistance and electro-motive force demanded by any given conditions is rather theoretical than practical. The resistance and electro-motive force of a battery continually change; the resistance may increase or diminish, the e.m.f. generally decreases as the battery is in active service. The oxidation of the hydrogen of the water molecule in a battery is termed depolarizing. This is an essential part of the action, and is often interfered with when a current is taken from a battery. In such case a battery is said to be polarized. Even standing on open circuit is liable to change a battery's constants. It follows that a calculation correct for a battery in good condition rapidly becomes incorrect, as the battery changes its constants as a current is taken from it or as it stands on open circuit. The following principles are adapted to calculating the cells of a battery of given constants required to maintain a given current through a given resistance. Let

n = number of cells in series.
r = resistance of a single cell.
e = e.m.f. of a single cell.
R = resistance of outer circuit.
I = current required.

The total e.m.f. of the circuit will be that of a single cell multiplied by the number of cells in series, which is equal to ne. The resistance of the battery is the resistance of a single cell multiplied by the number of cells if the cells are in series, or is nr. The total resistance of the circuit is the sum of the resistance of the battery and of that of the outer circuit, or is $nr + R$.

By Ohm's law the current is equal to the e.m.f. divided by the resistance, or

$$I = \frac{ne}{nr + R}. \tag{1}$$

Multiplying both members by $nr + R$ gives

$$nrI + RI = ne, \tag{2}$$

and transposing, $$RI = ne - nrI. \tag{3}$$

Dividing throughout by $e - rI$ and transposing,

$$n = \frac{RI}{e - rI}. \tag{4}$$

For this formula to be applicable to single cells in series it is evident that rI must be less in value than e. Otherwise the denominator will reduce to zero, giving infinity as the value of n, or in words stating that an infinite number of cells would be required unless the e.m.f. of a single cell exceeds the potential drop of a cell at the given current.

To meet this condition group two or more cells in parallel and treat each group as a single cell. The e.m.f. of the group is the same as that of a single cell; the resistance is that of a single cell divided by the number in the group. If m cells are grouped in parallel, we have e $(r/m \times I)$ for the denominator of the expression. If the value of this expression is twice the value of the required current, enough cells are in parallel to make it possible to obtain the current.

From the above considerations is found the smallest number of cells in parallel which will give a stated current. The minimum or smallest number of cells required is given if the internal resistance is equal to the external, carrying with it the condition that the e.m.f. of the battery, or ne, shall be twice the drop of the outer circuit. This drop is RI, so the condition is expressed in the equation

$$ne = 2\,RI. \tag{5}$$

Multiplying the second term of (4) by e and equating it with the second member of (5) gives

$$2\,RI = \frac{eRI}{e - rI}. \tag{6}$$

Multiplying by $e - Ir$ and dividing by $2\,RI$ gives

$$e - rI = \frac{e}{2}, \tag{7}$$

and transposing and reducing we find

$$rI = e - \frac{e}{2} = \frac{e}{2} \tag{8}$$

and

$$\frac{e}{r} = 2\,I, \tag{9}$$

which last equation expresses the condition for the smallest number of cells for a given current. The rule is thus expressed in words:

To obtain the smallest number of cells for a given current group enough cells in parallel to give on short circuit twice the required current. Treating the group as a single cell apply formula (4), when n will be the number of groups to be put in series. The total number of cells is the product of the number in a group by the number of groups in series.

Example. A current of 5 amperes is to be taken from a battery of cell constants 2 volts and $\frac{3}{10}$ ohm. The external resistance is 2 ohms. Calculate the number of cells required.

Solution. Applying (9) we have $\frac{e}{r} = 2 \div \frac{3}{10} = 6.66.$ As this is less than twice the current, the series arrangement will be disadvantageous, though possible, as requiring the greatest number of cells all in series and therefore giving the highest resistance. If two cells are placed in parallel the resistance of the group will be $\frac{3}{20}$ ohm, and $\frac{e}{r} = 2 \div \frac{3}{20} = 13.33,$ which is more than twice the current required. Treating this group as if it were a single cell and applying (4) we find for the groups in series

$$n = \frac{5 \times 2}{2 - (3/20 \times 5)} = \frac{10}{25/20} = 8.$$

The total number of cells is the number in parallel multiplied by the number in series, or $2 \times 8 = 16$.

Testing by Ohm's law, the resistance of the battery being

$\frac{3}{10} \times 8 \div 2 = \frac{24}{20}$ the resistance of the outer circuit being 2, and the e.m.f. of the battery being $2 \times 8 = 16$, the current is given by the formula, current $= \frac{16}{24/20 + 2} = 5$, which is the current required, thus proving the correctness of the operation.

Example. If possible arrange cells of the above constants in a series so as to give the same current through the same resistance.

Solution. Substituting the values of the problem in the denominator of (4) gives $e - rI = 2 - \left(\frac{3}{10} \times 5\right) = 0.5$. It is therefore possible to carry out the series connection. The number of cells, is by (4),

$$\frac{5 \times 2}{2 - (3/10 \times 5)} = \frac{10}{0.5} = 20.$$

Testing as before by Ohm's law we have

$$\text{current} = \frac{20 \times 2}{(20 \times 3/10) + 2} = \frac{40}{8} = 5,$$

proving the correctness of the operation.

The last arrangement is uneconomical in every respect. The two examples illustrate the advantage of proper connection of cells.

Example. Assume the figures of the last two problems, except that the resistance of the battery is $\frac{2}{5}$ ohm per cell. How many cells in series would give the current?

Solution. The denominator of the second term of (4) becomes $2 - \left(\frac{2}{5} \times 5\right) = 0$. Therefore an infinite number of cells would be needed. The current cannot be supplied by any number whatever of the cells in series.

PROBLEMS.

Calculate the e.m.f. and resistance of a battery of 486 cells 27 in series and 18 in parallel, each cell having 0.05 ohm resistance and 1.97 volt e.m.f. *Ans.* 0.075 ohm and 53.19 volts.

The constants of a battery cell are 1.8 volts and $\frac{1}{7}$ ohm. Arrange such cells for 3 amperes current through 21 ohms external resistance.
Ans. 70 cells in series and 2 in parallel.

Calculate a battery of cell constants 1 volt and 4 ohms to give a current of 5 amperes through a resistance of 5 ohms.
Ans. 50 cells in series and 40 in parallel.

Calculate the number of cells of 1.7 volt and 0.3 ohm each to maintain a current of 2.3 amperes through a resistance of 21 ohms with an efficiency of 75 per cent.

Ans. One group 2 in parallel and 28 in series, and another group 10 in series, the two in series with each other.

125 cells of 1.75 volts and $\frac{3}{4}$ ohm each are arranged 5 in parallel and 25 in series. There is an external resistance of 3.4 ohm. Calculate the battery constants and the current.
Ans. 43.75 volts; 3.75 ohms; 6.12 amperes.

Calculate the resistance of a battery to supply four magnets in parallel, the coil of each magnet being of 4.6 ohms resistance and the efficiency to be 90 per cent. *Ans.* 0.125 ohm.

Arrange cells of 1.1 volts and 6 ohms to supply above magnets with $2\frac{1}{4}$-ampere current. *Ans.* 2 cells in parallel will give 2.27 amperes.

What is the resistance and e.m.f. of a battery 11 in parallel and 7 in series with cell constants 1.5 volts and 0.25 ohm?
Ans. 0.159 ohm; 10.5 volts.

Calculate the same factors if the above battery is in parallel.
Ans. 0.00325 ohm; 1.5 volts.

Calculate the same for the battery in series.
Ans. 19.25 ohms; 115.5 volts.

What current will each of the arrangements give through 11 ohms external resistance? *Ans.* First arrangement, 0.94 ampere.
Second arrangement, 0.136 ampere.
Third arrangement, 3.8 amperes.

A current of 3 amperes is to be produced through a wire of 30 ohms resistance, with a battery of 2 volts and 0.2 ohm cell constants. Calculate the cells for 80 per cent efficiency.

Ans. Approximately 59 in series and 2 in parallel.

What is the exact efficiency of the above arrangement?

Ans. 83.3 per cent.

What current will be given by 51 cells, each cell of 1.07 volts 3 ohms, connected in series, through an external resistance of 19 ohms?

Ans. 0.317 ampere.

If 30 cells of the above battery are arranged 3 in parallel and 10 in series, what current will they give through the same resistance?

Ans. 0.369 ampere.

What will a single cell of the same battery give through the same resistance? *Ans.* 0.486 ampere.

NOTE.—This is a case where a single cell gives more current than a number of cells.

CHAPTER VIII.

ELECTRIC ENERGY AND POWER.

Potential. — Proof of Numerical Value of Potential. — Potential Drop or (see page 95) R. I. Drop. — Electric Energy. — Practical Unit of Electric Energy. — Electric Power or Activity. — Practical Unit of Electric Power. —Relations of Power to Current, Resistance, and e.m.f. — Equivalents of the Watt-Problems.

Potential. Potential is of various kinds, affecting mass, electric quantity, and other things, each kind of potential affecting only one of them. It can be defined as a condition of a point in space such that the energy involved in moving a unit mass or quantity from the point to an infinite distance or from an infinite distance to the point would be numerically equal to the potential.

The energy involved in the moving of a unit mass or the unit quantity from a point of one potential to that of another is numerically equal to the difference of the two potentials.

Electric potential affects electric quantity only.

Proof of Numerical Value of Potential. To prove the above, let e represent a charge of electricity at a point in space. Let a unit quantity be acted on by it at a distance r and also at a distance r', it having changed position from r to r'. The force exerted by e upon the unit quantity at these distances is e/r^2 and e/r'^2 (see page 33) respectively. The mean force is not the arithmetical average, because the force varies inversely with the square of the distance r. The mean force is the geometrical mean, which is the square root of the product of the two forces. This is $\sqrt{e/r^2 \times e/r'^2} = \sqrt{e^2/r^2r'^2} = e/rr'$. To get the energy this has to be multiplied by the path traversed; this we have taken as the distance from r to r', or as $r - r'$.

Performing the multiplication we have

$$\frac{e}{rr'}\times(r-r')=e\,\frac{r-r'}{rr'}=e\left(\frac{1}{r'}-\frac{1}{r}\right). \qquad (1)$$

If $r=\infty$, then energy $=e/r'$. (2)

As these expressions give the energy involved in the transfer of unit mass, they are the expressions (1) for the numerical value of the potential in the case of movement from r to r' and (2) in the case of movement from a point in space to an infinite distance. The latter has been defined as the value of absolute potential.

It is incorrect to say that potential is equal to the energy required to move a unit mass from or to an infinite distance to or from the place of potential, because potential and energy are not measured in the same unit. The number of potential units is equal to the number of energy units as described, but there is no equality of potential and energy.

Potential Drop or *RI* Drop. The current due to a definite e.m.f. is equal to the quotient of the e.m.f. less the *RI* drop in any part of the circuit divided by the resistance of the rest of the circuit. Calling the resistance of the part of the circuit which includes the *RI* drop R_1, this gives

$$I=\frac{E-R_1I}{R-R_1}.$$

The rule follows from Ohm's law. The e.m.f. expended on the part of resistance R_1 is by Ohm's law equal to R_1I. As the total e.m.f. expended on the system is E, the e.m.f. expended on the rest of the circuit is $E-R_1I$. The resistance of this portion, namely, the remainder of the circuit, is $R-R_1$. Hence by Ohm's law the current in the circuit is given by

$$I=\frac{E-R_1I}{R-R_1}\text{, as above.}$$

Example. The fall in potential, or *RI* drop, in a part of a circuit is $\frac{3}{4}$ volt while a current is passing due to an e.m.f. of 3 volts in the circuit. The resistance of the remainder of the circuit is 2 ohms. What is the intensity of the current?

Solution. Subtracting the *RI* drop of $\frac{3}{4}$ volt from the total e.m.f. of 3 volts gives the numerator of the fraction of the formulas $3 - \frac{3}{4} = 2\frac{1}{4}$. Dividing this by the resistance of the rest of the circuit, 2 ohms, gives

$$2\tfrac{1}{4} \div 2 = 1\tfrac{1}{8} \text{ amperes.}$$

Example. What is the resistance of the portion of the circuit in the above case in which the fall of potential of $\frac{3}{4}$ volt takes place?

Solution. By Ohm's law $R = E \div I$, and substituting, we have

$$R = \frac{3}{4} \div 1\tfrac{1}{8} = \frac{2}{3} \text{ ohm.}$$

Electric Energy. When electricity passes through a conductor it expends energy. The electric energy expended is converted into heat energy. The latter may be measured and used as the measure of electric energy.

If a unit of electric quantity actuated by a unit of e.m.f. passes through a conductor, a definite amount of heat will be produced. If two units of quantity actuated by one unit of e.m.f. pass through a conductor, twice the amount will be produced. If one unit of quantity as before but actuated by two units of e.m.f. passes, again twice the original amount will be produced. If two units of quantity actuated by two units of e.m.f. pass, then four times the original amount of heat will be produced.

The heat due to the passage of a quantity of electricity is the measure of the electric energy exerted. From the statements of the preceding paragraph it follows that the measure of the

electric energy incident to the passage of electricity through a conductor is the product of the units of quantity by the units of e.m.f. To bring about the conditions of the preceding paragraph, the resistance of the conductor or the time employed or both may be changed for the different cases.

Example. 30 C.G.S. units of quantity with 45 C.G.S. units of e.m.f. are expended on a conductor. How many ergs of energy are expended?

Solution. $30 \times 45 = 1{,}350$ ergs.

Practical Unit of Electric Energy. The practical unit of electric energy is the volt-coulomb, equal to 10^8 C.G.S. units of e.m.f. multiplied by 10^{-1} C.G.S. units of quantity, and therefore equal to 10^7 C.G.S. units of energy, or to 10^7 ergs, or to the joule. These absolute units are in the electro-magnetic system.

Example. 10^{10} absolute units of quantity at 10 units of e.m.f. are how many volt-coulombs?

Solution. $10^{10} \times 10 = 10^{11}$ ergs. 10^{11} ergs $\div 10^7 = 10^4$, or 10,000 volt-coulombs.

Example. If 31 coulombs at 20 volts are expended, how many C.G.S. units are expended?

Solution. 31 coulombs = 3.1 C.G.S. units. 20 volts = 20×10^8 C.G.S. units. $3.1 \times 20 \times 10^8 = 62 \times 10^8$ C.G.S. units.

Electric Power or Activity. The rate at which electrical quantity passes through a conductor is current strength or current intensity. One unit of quantity per second is unit current. The product of unit current by unit e.m.f. is unit rate of energy, unit activity, or unit power.

Practical Unit of Electric Power. The practical unit of power is the product of the practical unit of current by that of e.m.f. It is, as we have seen before, the volt-coulomb per second, the volt-ampere or watt. It is equal to 10^7 C.G.S. units of power.

Relations of Power to Current, Resistance, and e.m.f. It follows that, since the rate of energy varies with the product of e.m.f. by current, if either factor is constant the rate of energy will vary with the other one.

Example. Calculate the value of the watt in C.G.S. units.

Solution. The watt is the product of one volt by one ampere. One volt = 10^8 C.G.S. units. One ampere = 10^{-1} C.G.S. units. $10^8 \times 10^{-1} = 10^7$ C.G.S. units of rate of energy or of power. The C.G.S. unit of power is the erg per second. A rate of 10^7 ergs per second is one watt.

Example. A current of 16×10 C.G.S. units of current is actuated by 19×10^7 C.G.S. units of e.m.f. Calculate the value in watts.

Solution. $16 \times 19 = 304$; $10^7 \times 10 = 10^8$; the product of $16 \times 10 \times 19 \times 10^7 = 304 \times 10^8 = 3{,}040 \times 10^7$ C.G.S. units $= 3{,}040$ watts.

Example. Reduce and analyze 29 watts with 7 volts into C.G.S. units.

Solution. $29 \div 7 = 4.14$ amperes, the current in the case. 7 volts $= 7 \times 10^8$ C.G.S. units of e.m.f. 4.14 amperes $= 4.14 \times 10^{-1}$ C.G.S. units of current. The total is 29×10^7 C.G.S. units of power.

Example. A current of 3 amperes flows through a resistance of 3 ohms. What power is exerted?

Solution. By Ohm's law $E = IR$ the e.m.f. is 9 volts. The power is $9 \times 3 = 27$ volt-amperes or watts.

Example. 17 coulombs pass through a conductor of 7 ohms resistance in 15 seconds. Calculate the power.

Solution. A current of $17/15 = 1.133$ coulombs per second or amperes passes. The e.m.f. is given by Ohm's law $E = IR$; and substituting, $E = 1.133 \times 7 = 7.93$ volts. The power is therefore $7.93 \times 1.133 = 8.98$ watts.

By Ohm's law $E = IR$. Substituting this value of E in the expression of power or watts EI, we have

$$\text{Power} = I^2R,$$

by which expression power may be directly calculated from the resistance and current intensity.

Example. A current of 5 amperes flows through a resistance of 4 ohms. What is the power?

Solution. Substituting for I^2 and for R their values we have

$$\text{Power} = 25 \times 4 = 100 \text{ watts.}$$

Example. A current of 2×10 C.G.S. units flows through a resistance of 4×10^9 C.G.S. units. Calculate the power in watts.

Solution. Substituting as before for I^2, $(2 \times 10)^2 = 4 \times 10^2$, and for R, 4×10^9, we have

$$\begin{aligned}\text{Power} &= 4 \times 10^2 \times 4 \times 10^9 = 16 \times 10^{11} \\ &= 160{,}000 \times 10^7 \text{ C.G.S. units} = 160{,}000 \text{ watts.}\end{aligned}$$

By Ohm's law $I = E/R$. Substituting this value of I in the expression of power or watts EI, we have

$$\text{Power} = \frac{E^2}{R}.$$

Example. Assume 200 volts to act upon a resistance of 105 ohms. Calculate the energy absorbed per second.

Solution. Applying the expression E^2/R, we obtain for the power $(200)^2/105 = 381$ watts. In a second 381 watt-seconds or volt-coulombs are absorbed.

There are two expressions for electric power which include resistance as one of their component parts. These are I^2R and E^2/R.

Assuming the current to remain constant and the resistance to be doubled, the first expression becomes $I^2 \times 2\,R$, or $2\,I^2R$.

If the resistance is trebled the same expression becomes $3\,I^2R$, and so on. Therefore at constant current the electric power varies with the resistance.

Assuming the e.m.f. to remain constant, and doubling and trebling the resistance as before, the second expression becomes $E^2/2\,R$ and $E^2/3\,R$, whence it follows that at the same e.m.f. the electric power varies inversely with the resistance.

Example. The resistance of a lamp is 212 ohms. The conductor supplying it has a resistance of 4 ohms. How much energy is wasted on the conductor?

Solution. From the law of power distribution we have

Energy of conductor : energy of lamp : : 4 : 212.

The energy in the conductor is 4/212 of the energy in the lamp and $\dfrac{4}{4 + 212}$ of the energy in the circuit. These reduce to 1/53 and 1/54 approximately. The energy wasted is 1/53 that utilized in the lamp, and is 1/54 that consumed in the whole circuit.

It is correct to speak of energy or of power in this relation because the ratios are the same for both. This problem can be done usually by simple inspection.

Example. There are two lamps l and l' of resistances 211 and 214 ohms respectively. Calculate the relative energy which would be absorbed by each at the same e.m.f.

Solution. The proportion is given by the second law cited above,

$$l : l' : : 214 : 211$$

or the lamp of 211 ohms resistance absorbs 214/211 of the energy absorbed by the lamp of 214 ohms resistance.

Equivalents of the Watt. One watt-second is equal to 10^7 ergs. The electric energy expended in the passage of an electric current through a conductor is converted into heat energy. If the calorie is taken as equal to 4.185×10^7 ergs,

then to reduce watt-seconds to calories the number of watt-seconds must be divided by 4.185 or multiplied by 1/4.185 = 0.239, or as usually taken, 0.24.

Example. A current of 0.5 ampere is produced in a portion of a circuit by an e.m.f. of 110 volts. What number of calories is absorbed by this portion per second?

Solution. The watts are absorbed at the rate of $0.5 \times 110 = 55$ per second. The calories are $55 \times 0.24 = 13.2$.

Example. A current of 15.5 amperes at 110 volts is expended in an electric heater. How long will it take for it to boil a liter of water at 10° C. if 50 per cent of the heat is lost by waste?

Solution. $15.5 \times 110 = 1{,}705$ volt-amperes or watts. 1,705 watt-seconds multiplied by 0.24 gives 409.2 calories per second. A liter of water weighs 1,000 grams. To boil it, it has to be heated 90 degrees, making 90,000 calories that have to be imparted. $90{,}000 \div 409.2 = 220$ seconds, which is the time required if all the heat were utilized. But one half is lost, so that twice the time, or 440 seconds, is needed, or 7 minutes 20 seconds.

PROBLEMS.

What current will a 120-kilowatt generator give at 120 volts, running at full load? *Ans.* 1,000 amperes.

A current of 19 amperes actuated by 19 volts passes through a conductor for 8 minutes. Calculate the watt-seconds and ergs.
Ans. 173,280 watt-seconds; $17{,}328 \times 10^8$ ergs.

What is the rating of a generator that can deliver 320 amperes at 120 volts? *Ans.* 38.4 kilowatts.

Calculate the energy in 10^{19} C.G.S. units of quantity at 10 C.G.S. units of e.m.f. *Ans.* 10^{20} C.G.S. units; 10^{13} joules.

How many C.G.S. units and watts are there in 2 amperes at 9 volts?
Ans. 18×10^7 C.G.S. units; 18 watts.

In 10 joules of electric energy at 8 volts calculate the quantity in C.G.S. units and in coulombs.
Ans. 125×10^{-3} C.G.S. units; 1.25 coulombs.

How many joules are there in 32 C.G.S. units of quantity at 41 C.G.S. units of e.m.f.? *Ans.* $1,312 \times 10^{-7} = 0.000,131$ joules.

29 C.G.S. units of e.m.f. act on a resistance of 31 C.G.S. units. Calculate the power. *Ans.* 27.13 C.G.S. units of power.

If a resistance of (*a*) 41 is in series with one of (*b*) 39, what per cent of the total power if a current in passing through them will be expended on each? *Ans.* (*a*) 51.25 per cent; (*b*) 48.75 per cent.

A constant e.m.f. is maintained at the terminals of two lamps, one (*a*) of 120 ohms and one (*b*) of 115 ohms. What relative portion of the total energy will be absorbed by each?

Ans. (*a*) 48.9 per cent; (*b*) 51.1 per cent.

What power is expended in generating 29 coulombs at 19 volts in 11 minutes? *Ans.* 0.835 watts.

What is the energy developed in the above case? *Ans.* 551.1 joules.

What is the value of 11 watts in C.G.S. units?

Ans. $11 \times 10^7 = 110,000,000$ C.G.S. units.

Calculate the value of 16×10^4 C.G.S. units of current at 12×10^2 C.G.S. units of e.m.f.

Ans. 192×10^6 C.G.S. units of power; 19.2 watts.

If there are 31 watts of power at 32 volts, what is the current in amperes and C.G.S. units?

Ans. 0.969 amperes; 969×10^{-4} C.G.S. units.

171 coulombs are passed in 3 seconds through a resistance of 8 ohms. Calculate the watts. *Ans.* 25,992 watts.

The *RI* drop in one part of a circuit is 9 volts; the e.m.f. of the generator is 13 volts; the resistance of the rest of the circuit is 11 ohms. Calculate the current. *Ans.* 0.3636 ampere.

What is the total resistance in the above case? *Ans.* 35.8 ohms.

A number of lamps are using 51 amperes of current at 110 volts. Calculate the calories per second. *Ans.* 1,346.4 calories.

A conductor is assumed of constant resistance. Compare the power developed in it by the passage of currents of 7 and of 17 amperes.

Ans. 49 : 289 or as 10 : 59 (the first for the smaller current).

How many calories per second of energy are expended on 40 lamps each taking ½ an ampere of current with a drop of 110 volts?

Ans. 528 calories.

A current of 0.5 ampere is produced in a portion of a circuit by an e.m.f. of 110 volts. What number of calories is absorbed by this portion in a second? *Ans.* 13.20 calories.

Assume 200 volts acting upon a resistance of 105 ohms. Calculate the energy absorbed in one minute. *Ans.* 5,485.7 calories.

A current of 9 amperes passes through a resistance of 9 ohms. Calculate the calories per second. *Ans.* 175 calories.

How many 50-watt lamps absorb in one minute of operation the heat required to raise the temperature of 2 litres of water from 15° C. to 100° C.? *Ans.* 24 (exactly 23.6) lamps.

A current of 5 amperes is forced through a conductor by an e.m.f. of 6 volts. Calculate the power. *Ans.* 30 watts or volt-amperes.

A current of 5 amperes is actuated by a voltage of 4 volts. How many calories does it develop in one hour? *Ans.* 17,280 calories.

379 calories are developed in a conductor of 3 ohms resistance. Determine the joules. *Ans.* 1,579 joules.

If in the above example 15 seconds time was expended in the operation, what was the current and e.m.f.? *Ans.* 5.91 amperes.
17.77 volts.

A potential difference of 31 volts is maintained between the ends of a conductor of 13 ohms resistance. How many calories are developed in 3 minutes? *Ans.* 3,193.5 calories.

25 absolute units of e.m.f. cause a current of 75 absolute units to flow through a conductor. How many ergs per second of energy are developed? *Ans.* 1,875 ergs.

Calculate the ergs in 95 C.G.S. units of quantity actuated by 19 C.G.S. units of e.m.f. *Ans.* 1,805 ergs.

10^3 C.G.S. units of current are maintained by 10^4 C.G.S. units of e.m.f. What is the power? *Ans.* 10^7 ergs per second = 1 watt.

Between the terminals of a conductor in which 11,917,250 ergs per second are developed a potential difference of 3×10^8 C.G.S. units of e.m.f. is maintained. What is the current?

Ans. 0.0397 C.G.S. units.
0.397 amperes.

How many C.G.S. units of current and how many amperes are there in an electric horse-power at 75 volts? *Ans.* 0.995 C.G.S. units.
9.95 amperes.

An e.m.f. of 5 volts acts upon a resistance of 7 ohms. Calculate the power. *Ans.* 3.55 watts or volt-amperes.

A current of 27 amperes passes through a conductor of 13 ohms resistance. An additional resistance of 17 ohms is added to the conductor, in series therewith, and then one of 29 ohms, also in series. Calculate the power expended in the three cases if the same current is maintained. *Ans.* 9,477 watts.
21,870 watts.
43,011 watts.

A uniform current is maintained through three conductors in parallel, each conductor being of identical resistance, and afterwards through the same three conductors in series. Compare the power expended in each case.

Ans. 1 : 9, or as the square of the resistances.

A current of 6 amperes flows through 12 lamps in parallel; each lamp is of 220 ohms resistance. How many watts are expended, and in 2 hours' running how many joules and how many ergs?

Ans. 660 watts.
4,752,000 joules.
$4{,}752 \times 10^{10}$ ergs.

How many ergs are developed per second in the operation of a generator delivering 21 amperes at 118 volts potential?

Ans. $2{,}478 \times 10^{7}$ ergs per second.

What is the power of the above generator?

Ans. 2,478 watts, or nearly 2½ kilowatts.

The armature of a shunt-wound dynamo has a resistance from brush to brush of 0.03 ohm; the e.m.f. at the brushes is 110 volts, with a current of 150 amperes in the outer circuit; the resistance of the shunt field is 55 ohms. What is the armature loss in watts? In a shunt-wound dynamo the outer circuit and shunt are in parallel. Give general data.

Ans. Resistance of outer circuit 0.733 ohm.
Combined resistance of outer circuit and shunt 0.7235 ohm.
Current in armature 152 amperes.
Armature loss 693 watts.

CHAPTER IX.

BASES AND RELATIONS OF ELECTRIC UNITS.

Effects of an Electric Current. — Two Systems of C.G.S. Electric Units. — The Basis of Practical Units. — The Basis for Measurement and Definition of a Current. — The Absolute C.G.S. Electro-magnetic Unit of Current. — The Tangent Compass. — Action of Earth's Field. — Angle of Divergence. — Combined action of Coil and Earth's Field on Magnet (See page 109). — Tangent Galvanometer Formula. — Determining C.G.S. units of e.m.f. and Resistance. — The Absolute C.G.S. Electrostatic Unit of Quantity. — Determination of the E.S. Unit of Potential. — The Attracted Disk Electrometer. — Other Units of the Electrostatic System. — The E.M. and E.S. System of Units. — Equivalents of the Two Systems of Units. — Derivation of Practical Units from Absolute Units. — Reduction Factor. — Problems. — Dimensions of E.M. Quantities. — Dimensions of Magnetic Quantity. — Dimensions of Current in E.M. System. — Dimensions of Electric Quantity in E.M. System. — Dimensions of Potential in E.M. System. — Dimensions of Resistance in E.M. System. — Dimensions of Capacity in E.M. System. — Dimensions of Electric Quantity in E.S. System. — Dimensions of Surface Density in E.S. System. — Dimensions of Potential in E.S. System. — Dimensions of Capacity in E.S. System. — Dimensions of Current in E.S. System. — Dimensions of Resistance in E.S. System. — Dimensions of Magnetic Quantity. — Dimensions of Surface Density of Magnetism. — Dimensions of Magnetic Intensity. — Dimensions of Magnetic Potential. — Dimensions of Magnetic Power. — Dimensions of Electric Intensity in E.S. System.

Effects of an Electric Current. Three things must co-exist in an electric circuit — electro-motive force, current, and resistance. The effects of an electric current depend upon the rate of electric quantity which constitutes it. If any two of the above are known the other can be determined by Ohm's law.

Two Systems of C.G.S. Electric Units. There are two systems of electric units of the C.G.S. system. They are termed absolute units. One is the absolute system of electro-static units; the other is the absolute system of electro-magnetic units.

The Basis of Practical Units. The absolute electro-magnetic units are usually assumed to be the basis of the practi-

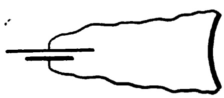
cal units such as the volt and ampere, although the latter could be derived from the electrostatic units.

The Basis for Measurement and Definition of a Current. If a current passes through a conductor it produces various effects. These effects by their intensity give a basis for measuring and defining the intensity of the current. One of these effects is the production of a field of force. A magnet pole is acted on by a field of force; it is attracted or repelled by a current. As the effect diminishes rapidly with the distance intervening between current and magnet, the effect is practically limited in extent to a small volume of space. The volume within which the effect is discernible is called a field of force. Theoretically a field of force may be as large as the universe. Practically artificial fields of force are small in volume. The field of force of the field magnet of a bipolar dynamo is little more in extent than the volume existing between its pole faces.

The intensity of a field of force can be measured by its action on a magnet pole of known strength.

A unit magnet pole is one of such strength that it will attract or repel another unit pole at a distance of one centimeter with a force of one dyne.

The Absolute C.G.S. Electro-magnetic Unit of Current. The absolute C.G.S. unit of current is the following. It is



the intensity of a current which passing through a conductor 1 centimeter long bent into the arc of a circle of 1 centimeter radius will attract or repel with a force of 1 dyne a unit magnet pole placed at the center of such circle.

The diagram shows an arc of this radius with its center indicated. Calling force f we have for the case described

$$f = 1. \tag{1}$$

Suppose that the current goes through a complete circle of 1 centimeter radius. Its action on a magnet pole at its center

will be greater than in the last case in the ratio of the lengths of the conductors. A circle of radius 1 has a length or circumference of 2π. The ratio of the arc of the first case to that of the circle of the second case is $1 : 2\pi$. Therefore as these are the lengths of the conductors acting on the magnet pole, and as the intervening distance is the same in both cases, the force acting on the magnet is expressed by

$$f = 2\pi. \tag{2}$$

Example. With what force will a unit magnet pole at the center of a circular conductor, the circle into which it is bent being of 1 cm. radius, be acted on when a unit current passes?

Solution. Substituting for π in (2) its value, 3.1416, we have for the force

$$f = 2\pi = 2 \times 3.1416 = 6.2832 \text{ dynes}.$$

Suppose the wire is bent into a circle of radius r. The length of the conductor is now $2\pi r$, and as far as this change of length of conductor is concerned its action on the pole at its center varies directly with the length of the conductor acting on it. But the distance of the conductor from the pole is changed in direct ratio with r. As radiant action, which is the action exerted on the pole in the cases under consideration, varies inversely with the square of the distance, we have to express this action also in the next formula. The direct ratio of the length of the conductor and the inverse ratio of the square of the distance give the formula

$$f = \frac{2\pi r}{r^2} = \frac{2\pi}{r}. \tag{3}$$

Example. Assume the radius of a conductor bent into a circle to be 7 cm. With what force will a unit current passing through it act upon a unit pole at its center?

Solution. Substituting in (3) the values of r and of π we have

$$f = \frac{2 \times 3.1416}{7} = \frac{6.2832}{7} = 0.8976 \text{ dyne}.$$

Suppose that the circular conductor is wound around the circle a number of times n, it then obviously acts with n times the intensity of a single turn. Equation (3) then becomes

$$f = \frac{2\pi n}{r}, \tag{4}$$

and if the current instead of being a unit current is of strength i, equation (4) becomes

$$f = \frac{2\pi n i}{r}, \tag{5}$$

whence by division

$$i = \frac{fr}{2\pi n}. \tag{6}$$

If the magnet pole is of strength M the force exerted upon it will be M times as great as that exercised upon a unit pole. This is because the force exercised is a radiant force between two things, and such force is equal to the product of the two forces. Equations (5) and (6) for a magnet pole of strength M become

$$f = \frac{2\pi n i M}{r}. \tag{7}$$

$$i = \frac{fr}{2\pi n M}. \tag{8}$$

Example. A magnet pole of 5 units strength at the center of a circle of 11 cm. radius composed of 24 turns of wire is acted on by a force of 377 dynes. What is the intensity of current passing through the wire?

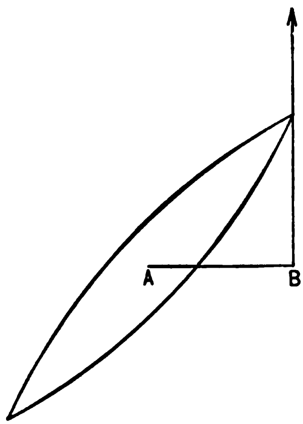
Solution. By substituting in formula (8) we obtain

$$i = \frac{377 \times 11}{6.2832 \times 24 \times 5} = 5.5 \text{ C.G.S. units.}$$

The Tangent Compass. If in the center of such a coil a very short compass needle is established a tangent compass is

produced. The compass needle is acted on by the horizontal component of the earth's magnetic field. The circle of the galvanometer is placed in the magnetic meridian approximately north and south. A current passing through the wire tends to deflect the needle into a position at right angles to the plane of the coil. This is at right angles to the position in which the earth's component tends to keep it.

Action of Earth's Field. The earth's field acts upon the needle with the product of the horizontal component of the earth's magnetism by the strength of the magnet. This product we may call HM, H representing the horizontal component of the earth's magnetism and M the force of the magnet.



Angle of Divergence. At a given angle of divergence θ from the magnetic meridian the lever arm of the earth's action on the magnet is equal to the product of the magnet's length by the sine of the angle of divergence. This is shown in the diagram. The earth acts upon both arms of the magnet with virtual lever arms of length AB, and the sum of the two is as if it acted upon the magnet with a single lever arm equal to the length of the magnet multiplied by sin θ.

By the same course of reasoning we find that the coil acts with a lever arm equal to the product of the magnet's length by the cosine of the angle of divergence.

Combined Action of Coil and Earth's Field on Magnet. Calling the length of the magnet l we have for the earth's action $HMl \sin\theta$, and for the action of the coil $\frac{2\pi niMl\cos\theta}{r}$.

In any position of equilibrium which the needle assumes it will be acted on equally by these two opposing forces. They therefore may be made equal to each other, or

$$\frac{2\,\pi niMl\cos\theta}{r} = HMl\sin\theta, \tag{9}$$

in which M and l eliminate each other and which by transposition gives

$$i = \frac{r}{2\,\pi n} H \frac{\sin\theta}{\cos\theta} = \frac{rH\tan\theta}{2\,\pi n}. \tag{10}$$

Tangent Galvanometer Formula. This is the formula of the tangent galvanometer, developed here at length to show how the electro-magnetic C.G.S. unit of current can be directly determined. It will be observed that the only constant entering into the formula is the horizontal component of the earth's magnetism. This has been determined for a great many places.

Example. The radius of a coil is 30 cm., the number of turns is 5, the value of H is 0.1590 dyne, and the needle is deflected by a current passing through the coil to an angle of 50° 11′. Calculate the current strength in C.G.S. units.

Solution. Substituting in equation (10) we have

$$i = \frac{30 \times 0.1590}{6.2832 \times 5} \times \tan 50^\circ\, 11' \; (= 1.2) = 0.18 \text{ C.G.S. unit}$$

of current.

Determining C.G.S. Units of e.m.f. and Resistance. It now remains to be shown how the C.G.S. units of e.m.f. and of resistance are found.

The energy due to an electric discharge is equal to the product of the quantity discharged by the potential difference of the points or places between which the discharge takes place. The ergs per second of energy expended when a current of electricity is passing through a conductor is equal to the current strength multiplied by the e.m.f. required to maintain the cur-

rent through the conductor in question. The energy expended is converted into heat energy. All units are assumed as of the C.G.S. system.

By placing a conductor in a calorimeter, passing a known current through it, the current being taken in C.G.S. units, determining the ergs per second produced, the e.m.f. expended on the conductor is manifestly equal to the ergs divided by the current because the current is numerically equal to the quantity per second. Then having the current and e.m.f., the resistance of the conductor is calculated by Ohm's law.

Example. A current of 0.114 C.G.S. units passing through a calorimeter wire for one minute produces 40 calories of heat. Calculate the e.m.f. between the terminals of the wire and the resistance of the wire.

Solution. To reduce the calories to ergs they must be multiplied by 416.7×10^5, whence the ergs per minute corresponding to the calories of the problem are $40 \times (416.7 \times 10^5) = 1{,}666.7 \times 10^6$ ergs $= 1{,}666{,}700{,}000$ ergs. Dividing the ergs per minute given above by 60 gives ergs per second. $(1{,}666.7 \times 10^6) \div 60 = 278 \times 10^5$ ergs per second. This is to be divided by the quantity of electricity per second, which is numerically equal to the current. $(278 \times 10^5) \div 0.114 = 243 \times 10^6$ C.G.S. units of e.m.f. The resistance of the wire is calculated by Ohm's law, $R = E/I$, whence $(243 \times 10^6) \div 0.114 = 213 \times 10^7$ C.G.S. units of resistance.

1 C.G.S. unit of current = 10 amperes. The current therefore is $0.114 \times 10 = 1.14$ ampere.

10^8 C.G.S. units of e.m.f. = 1 volt. The e.m.f. therefore is $(243 \times 10^6) \div 10^8 = 2.43$ volts.

10^9 C.G.S. units of resistance = 1 ohm. The resistance therefore is $(213 \times 10^7) \div 10^9 = 2.13$ ohms.

This section is designed to show the relation of the absolute units to the practical ones and also the relation of the apparently

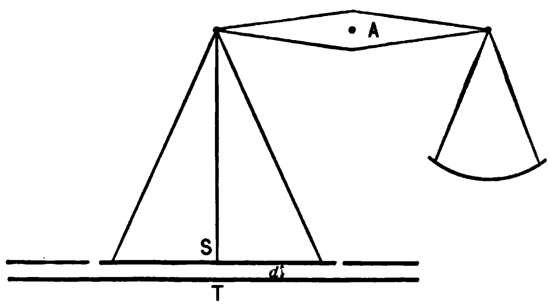
abstract bases of the absolute system to the standards of the engineer. It is not to be taken as showing approved details of making these determinations, which exact the most accurate work and refinement in methods.

An ammeter and a voltmeter could be standardized or calibrated by the above method; the length, diameter, and material of a wire of 1 ohm resistance could be determined, and with these bases the other practical units could be determined and standards fixed for them.

The Absolute C.G.S. Electrostatic Unit of Quantity. The absolute electrostatic unit of quantity is that quantity which would attract or repel a similar quantity one centimeter distant with a force of one dyne. From this as a basis a full series of units of the E.S. system, as the term may be conveniently abbreviated, is deduced.

Determination of the E.S. Unit of Potential. To obtain an experimental basis we may start with the determination of potential difference. The attracted disk electrometer can be used for this determination.

The Attracted Disk Electrometer. It comprises a disk held above and parallel to a plate, the plate being much larger



than the disk. The apparatus is so arranged that the attraction between plate and disk and the distance separating them can be

determined. In the diagram the disk is represented by the line S, the plate by the line T, and an annular plate surrounding the disk, as close to it as possible, is also indicated.

By touching one of the two, disk or plate, with an excited conductor, a charge is imparted and they attract each other. Call the surface density of the charge + for one surface and − for the other.

The attraction of an indefinitely large plane with a surface density σ for a point, in this case a unit of quantity, placed opposite its center, is $2\pi\sigma$ (see page 270). This is the attraction of the plate for a unit of quantity of electricity on the disk. Surface density is the quantity of electricity per unit area of the plate or disk.

Call the area of the disk S. Then the entire quantity on its surface is $S\sigma$. It will be seen that σ represents the quantity on a unit area of the disk, because the surface densities of the charges on plate and disk are identical. The attraction A between disk and plate is therefore

$$A = 2\pi\sigma \times S\sigma = 2\pi S\sigma^2, \tag{1}$$

because $S\sigma$ is numerically equal to the number of units of quantity on the disk.

Transposing (1) gives

$$\sigma^2 = \frac{A}{2\pi S}, \tag{2}$$

If a point is situated between the disk and plate, both will act, the one attracting and the other repelling any mass between them, and hence both exercising force in the same direction. The force at such a point will be the force exercised by one of the surfaces multiplied by 2, or $2\pi\sigma \times 2, = 4\pi\sigma$.

If a unit quantity is transferred from one plane to the other, as from disk to plate, the energy involved is numerically equal to the difference of potential between the two surfaces. This

is because the product of potential difference by quantity is energy, and as the quantity transferred is supposed to be equal to unity, the potential difference and the energy have the same numerical value. Call the potential difference between the two surfaces P and the distance between the plates d. Then the potential difference will be equal to the force $4\pi\sigma$ exercised on a unit of quantity multiplied by the distance of transfer d, giving

$$P = 4\pi\sigma d. \tag{3}$$

Substituting for σ in this equation its value from (2) we have

$$P = 4\pi d \times \sqrt{\frac{A}{2\pi S}} = d\sqrt{\frac{8\pi A}{S}}.$$

The force of attraction A is measured, the area of the disk S and the distance d are known. Substituting for A, S, and d their values, the numerical value in electrostatic units of the potential difference between the plate and disk is found.

In carrying out this calculation, C.G.S. units must be used. The force of attraction must be expressed in dynes.

Example. In an attracted disk electrometer, the area of the disk was 10 square cm. The distance from the disk to the plate was 0.5 cm. After imparting a charge to the disk the attraction was 0.092 gram. Calculate the potential difference in electrostatic C.G.S. units.

Solution. Substituting for the symbols of the formula (4) their values it becomes

$$P = 0.5\sqrt{\frac{8\pi \times (0.092 \times 981)}{10}} = 7.53 \text{ electrostatic units.}$$

The part of the expression in parenthesis (0.092×981) is the force of attraction in dynes, on the assumption that the force of gravity was measured by an acceleration of 981 cm. It is equal to 90.252 dynes.

Other Units of the Electrostatic System. If one such unit of e.m.f. were expended on maintaining a current of such strength that the two would expend 1 erg of energy per second, the current would be of one electrostatic absolute unit strength. The resistance of the conductor through which this current would be forced by a unit of potential difference would be one electrostatic unit in amount.

From this basis the whole series of electrostatic units can be deduced.

The E.M. and E.S. Systems of Units. The value of a current or of potential difference may be determined in absolute units by various methods which give it directly. Examples of such are given in the preceding pages, where they are employed simply to give the concrete idea of what these units are, of how they are related to the three fundamental units of time, space, and mass, and of how they can be determined experimentally by direct use of the centimeter, gram, and second. One set of electric units is based on the attraction of oppositely charged surfaces for each other, or, what is the same thing, on the repulsion of similarly charged ones. This set of units is called the electrostatic series. Another series is based on the action of an electric current on a magnet pole. This series is known as the electro-magnetic series. Both sets of units are C.G.S. units and can be used for electric specification and calculation, one just as well as the other.

Equivalents of the Two Systems of Units. The units of the two systems have values which differ from the values of corresponding units of the other system. The relation of the values has been determined by experiment, and the results are very accurate, yet not absolutely so.

The E.M. absolute unit of quantity is equal to 3×10^{10} E.S. units. The coulomb is equal to 10^{-1} E.M. absolute units, and therefore is equal to 3×10^{9} E.S. units.

Exactly the same relations exist for the current units, so that the ampere is equal to 3×10^9 E.S. units.

The E.S. unit of potential is equal to 3×10^{10} E.M. absolute units. The volt is equal to 10^8 E.M. absolute units, and therefore is equal to $\frac{1}{3} \times 10^{-2}$ E.S. unit.

The E.S. unit of resistance is equal to 9×10^{20} E.M. absolute units. The ohm is equal to 10^9 E.M. absolute units, and therefore is equal to $\frac{1}{9} \times 10^{-11}$ E.S. unit.

The E.M. absolute unit of capacity is equal to 9×10^{20} E.S. units. The practical unit of capacity, the farad, is equal to 10^{-9} E.M. absolute units, and therefore is equal to 9×10^{11} E.S. units. The subject of capacity is treated of in Chapter XV.

Derivation of Practical Units from Absolute Units. The practical units are derived from the absolute units by either of two methods, each giving an identical result. The simplest method is to reduce the units of the absolute series to those of the practical series by multiplying by defined factors. The table gives the value of the units of the practical series in terms of the absolute units, with the factors. The factors are all powers of 10 or multiples thereof.

Practical Units.	E.M. Absolute Units.	E.S. Absolute Units.
Coulomb	10^{-1}	3×10^9
Ampere	10^{-1}	3×10^9
Farad	10^{-9}	9×10^{11}
Volt	10^8	$\frac{1}{3} \times 10^{-2}$
Ohm	10^9	$\frac{1}{9} \times 10^{-11}$

The relations of the absolute units are given in the following table.

	E.M. Absolute Units.	E.S. Absolute Units.	E.S. Absolute Units.	E.M. Absolute Units.
Quantity	1	3×10^{10}	1	$\frac{1}{3} \times 10^{-10}$
Current	1	3×10^{10}	1	$\frac{1}{3} \times 10^{-10}$
Capacity	1	9×10^{20}	1	$\frac{1}{9} \times 10^{-20}$
Potential	1	$\frac{1}{3} \times 10^{-10}$	1	3×10^{10}
Resistance	1	$\frac{1}{9} \times 10^{-20}$	1	9×10^{20}

Example. Calculate the equivalent of 5 volts in electrostatic units (absolute).

Solution. A volt is equal to 10^8 absolute E.M. units; 5 volts therefore are equal to 5×10^8 absolute E.M. units. For the equivalent electrostatic units this must be divided by 3×10^{10}. Then

$$5 \times 10^8 \div 3 \times 10^{10} = \frac{5}{3} \times 10^{-2} = 0.0166 \text{ E.S. units.}$$

Or by direct process, using the equivalent of the table, $5 \times \frac{1}{3} \times 10^{-2} = 0.0166$ E.S. units.

Example. What is the value of the volt in absolute electrostatic units?

Solution. From the above calculation it follows that the value of the volt is $0.0166 \div 5 = 0.0033$ E.S. unit.

Or by the table, $1 \times \frac{1}{3} \times 10^{-2} = 0.0033$ E.S. unit.

Example. What is the value of 28.3 E.S. absolute units in volts?

Solution. As the E.M. unit is equal to the E.S. unit divided by 3×10^{10}, the reverse rule holds and the E.S. unit is equal to the E.M. unit multiplied by 3×10^{10}. This gives

$$28.3 \times 3 \times 10^{10} = 849 \times 10^9 \text{ absolute E.M. units.}$$

To reduce this to volts we must divide by the equivalent of the volt in absolute E.M. units, which is 10^8.

$$(849 \times 10^9) \div 10^8 = 8{,}490 \text{ volts.}$$

Or by the table, $28.3 \times 3 \times 10^2 = 8{,}490$ volts.

Example. An e.m.f. of 120 volts produces a current of 5 amperes. Reduce these quantities to E.S. absolute units; apply Ohm's law to the result, thus obtaining the value of the resistance in the same system of units; reduce the resistance thus calculated to ohms and test the correctness of the operation by Ohm's law directly applied.

Solution. The absolute value of 120 volts in the E.M. system is $120 \times 10^8 = 12 \times 10^9$. Applying the equivalent to obtain the value of this quantity in absolute E.S. units we have

$$12 \times 10^9 \div (3 \times 10^{10}) = 4 \times 10^{-1} = \text{e.m.f. E.S.}$$

5 amperes are equal to 5×10^{-1} absolute E.M. units. Applying the equivalent we find

$$5 \times 10^{-1} \times (3 \times 10^{10}) = 15 \times 10^9 = \text{current in E.S. units.}$$

Applying Ohm's law to the E.S. units thus determined gives

$$4 \times 10^{-1} \div (15 \times 10^9) = 266 \times 10^{-13} = \text{resistance E.S.}$$

To test the correctness of the operation proceed thus: The original data give the resistance by Ohm's law as $E/I = 120 \div 5 = 24$ ohms.

24 ohms $= 24 \times 10^9$ absolute E.M. units. Applying the equivalent we find

$24 \times 10^9 \div (9 \times 10^{20}) = 266 \times 10^{13}$, as found by the former operation. This goes to prove the correctness of the work. This example is only given as an exercise. The results are better obtained by the use of the tables.

The practical units of the E.M. system can be directly derived from the dimensional formulas of units by changing the units of length and mass.

Let the unit of mass be 10^{-11} gram and let the unit of length be 10^9 cm., the unit of time being the second, as in the absolute system. If these are introduced into the E.M. dimensional formulas of units the resulting units will be of the practical system, volts, amperes, ohms, and the regular engineering units.

Example. Calculate the value of the ampere in absolute electro-magnetic units.

Solution. The dimensions of the unit are $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$. Substituting for these symbols their values as given above we have for the ampere

$$10^{-\frac{11}{2}} \times 10^{\frac{9}{2}} \times 1^{-1} = 10^{-1} \text{ C.G.S. units.}$$

Ten amperes are equal to one absolute electro-magnetic unit.

Example. Calculate the value of the ohm and volt as above.

Solution. For the ohm the dimensions are LT^{-1}, and for the volt $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$. Substituting as above gives

$$\text{Ohm} = 10^{9} \times 1^{-1} = 10^{9} \text{ C.G.S. units.}$$
$$\text{Volt} = 10^{-\frac{11}{2}} \times 10^{\frac{27}{2}} \times 1^{-2} = 10^{8} \text{ C.G.S. units.}$$

Reduction Factor. The value of the reduction factor of the two systems of C.G.S. units, which is also the numerical value of the velocity of light in centimeters, is about 3×10^{10}, a fact applying to the electro-magnetic theory of light.

The reduction factor has been determined with close approximation to the exact figure by several investigators with results which are reasonably concordant. The following are some of the results reached by recent observers.

1883, J. J. Thomson	2.963×10^{10}
1889, Lord Kelvin	3.004×10^{10}
1890, J. J. Thomson and G. F. C. Searle	2.9955×10^{10}
——, O. Lodge and Glazebroke	3.009×10^{10}

Example. Calculate the dimensions of the reduction factor to reduce the E.S. unit of capacity to the E.M. unit.

Solution. The dimensions of the E.S. unit are L; of the E.M. unit, $L^{-1}T^{2}$. To find the factor the second must be divided by the first, thus:

$L^{-1}T^{2} \div L = L^{-2}T^{2}$, which is the reciprocal of velocity squared.

Example. Make the same calculation for current units.

Solution. The dimensions of the E.S. unit are $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$; of the E.M. unit, $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$. Dividing as above we find $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} \div M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} = LT^{-1}$, which are the dimensions of velocity.

Dimensions of E.M. Quantities. Two magnet poles of equal strength and of opposite polarity attract each other in direct proportion to the square of the quantity of magnetism in one of the poles and in inverse proportion to the square of the distance between them.

Dimensions of Magnetic Quantity. Call the magnetism of each pole Q, and let the distance separating them be denoted by L. The attraction of the poles is a force, and we have seen that the dimensions of a force are MLT^{-2}. The force in the particular case is, as just indicated, Q^2L^{-2}. Putting these two equal to each other, as they are the same, we have

$$Q^2L^{-2} = MLT^{-2}, \text{ whence } Q^2 = ML^3T^{-2} \text{ and}$$
$$Q = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1},$$

which are the dimensions of magnetic quantity. Care must be taken not to confuse magnetic quantity with electric quantity. They are totally distinct.

Dimensions of Current in E.M. System. Let I denote the strength of a current passing through a conductor of length L and bent into an arc of a circle of radius L as explained. Let a magnet pole representing a quantity of magnetism Q be at the center of the circle of which the conductor is an arc. The force exercised between the current and magnet pole will follow the laws of radiant action, with the additional law that the action will vary directly with the length of the conductor. The action will be therefore the product of the current strength by the length of the conductor by the quantity of magnetism, the whole divided by the square of the distance separating the two loci of force, namely, the conductor and the

magnet pole. This gives the expression for the action between the two,

$$ILQ \div L^2 = IQL^{-1}.$$

As this action is force, it can be put equal to the dimensions of force, or

$$IQL^{-1} = MLT^{-2}.$$

Q is magnetic quantity, whose dimensions have just been determined and are $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$. Substituting these for Q in the last equation gives

$$I\,(M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1})\,L^{-1} = I\,(M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}) = MLT^{-2}, \text{ whence}$$
$$I = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1},$$

which are the dimensions of electric current in the electromagnetic system.

Dimensions of Electric Quantity in E.M. System. The quantity of electricity passed by a current in any given time is evidently equal to the product of the current by the time during which it is passing, as it is clear that a given current will pass twice as great a quantity in two minutes that it will in a single minute. Multiplying the dimensions of current by the dimensions of time T gives the dimensions of electric quantity, $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} \times T$, which reduces to

$$M^{\frac{1}{2}}L^{\frac{1}{2}}.$$

These dimensions are quite different from those of magnetic quantity.

Dimensions of Potential in E.M. System. The product of e.m.f. by electric quantity is energy. Therefore if the dimensions of energy ML^2T^{-2} are divided by those of electric quantity $M^{\frac{1}{2}}L^{\frac{1}{2}}$ the dimensions of e.m.f. will be given, or

$$ML^2T^{-2} \div M^{\frac{1}{2}}L^{\frac{1}{2}} = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}.$$

Dimensions of Resistance in E.M. System. According

to Ohm's law resistance is equal to the quotient of e.m.f. divided by current strength. Carrying out this division with the dimensions as just determined gives the dimensions of resistance, thus:

$$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} \div M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} = LT^{-1}.$$

These are also the dimensions of velocity or rate of motion.

Dimensions of Capacity in E.M. System. The capacity of a surface, as of a condenser, is defined as determined by the quantity of electricity required to raise its potential a given amount. Its dimensions are therefore equal to the dimensions of quantity $M^{\frac{1}{2}}L^{\frac{1}{2}}$ divided by the dimensions of e.m.f. $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$, or $M^{\frac{1}{2}}L^{\frac{1}{2}} \div M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} = L^{-1}T^{2}$.

The dimensions of electric energy, so called, are the same as those of every other form of energy, $ML^{2}T^{-2}$.

The same applies to electric power, whose dimensions are $ML^{2}T^{-3}$.

Dimensions of Electric Quantity in E.S. System. A quantity of electricity acts upon another equal quantity at a distance d with force, following the laws of radiant action, and such force varies accordingly directly with the square of the quantity q and inversely with the square of the distance d. Thus calling force f we have $f = q^2/d^2 = q^2d^{-2}$, whence $q^2 = d^2f$ and $q = d\sqrt{f} = df^{\frac{1}{2}}$. Substituting for d its dimension L and for f its dimensions, those of force, MLT^{-2}, gives

$$q = L \times \sqrt{MLT^{-2}}, = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1},$$

the dimensions of electric quantity.

Dimensions of Surface Density in E.S. System. Surface density is the quantity per unit area; L^2 is the expression for unit area. Dividing the dimensions of quantity as just determined, $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$, by L^2 gives

$$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \div L^{2} = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1},$$

the dimensions of surface density.

Dimensions of Potential in E.S. System. Electrostatic potential, which is e.m.f. in the electrostatic system, is energy ML^2T^{-2} divided by quantity $M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$, or e.m.f. $= ML^2T^{-2} \div M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$.

Dimensions of Capacity in E.S. System. Capacity is quantity of electricity per unit of e.m.f., as in the electro-magnetic system. This gives

$$k = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \div M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} = L.$$

Thus electrostatic capacity can be expressed as a length.

Dimensions of Current in E.S. System. Current is quantity per unit time, giving

$$i = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \div T = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}.$$

Dimensions of Electric Intensity in E.S. System. Electric intensity is force per unit of electric quantity ($M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$), or

$$h = MLT^{-2}\text{ (force)} \div M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}.$$

Dimensions of Resistance in E.S. System. Resistance is given by Ohm's law as e.m.f. divided by current, or

$$r = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} \div M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} = L^{-1}T.$$

Dimensions of Magnetic Quantity. Quantity of magnetism is deduced from the law of the tangent galvanometer, as it may be referred to for convenience, given on page 110. A quantity q^1 of magnetism at the center of a circular current of intensity i, of radius r, and of length l is acted on by a force as explained in the section referred to, which force is given by the expression $f = ilq' \div r^2$, whence

$$q' = fr^2 \div il,$$

and as $f = MLT^{-2}$, $r^2 = L^2$, $i = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$, and $l = L$, we find by substituting these values

$$q' = (MLT^{-2} \times L^2) \div (M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} \times L) = M^{\frac{1}{2}}L^{\frac{1}{2}},$$

the dimensions of quantity of magnetism.

Dimensions of Surface Density of Magnetism. Surface density of magnetism is the quotient of the above expression divided by unit area L^2, or

$$\sigma = M^{\frac{1}{2}}L^{\frac{3}{2}} \div L^2 = M^{\frac{1}{2}}L^{-\frac{1}{2}}.$$

Dimensions of Magnetic Intensity. Magnetic intensity is force per unit of quantity, $L^{\frac{3}{2}}M^{\frac{1}{2}}$. This gives

$$h' = f \div M^{\frac{1}{2}}L^{\frac{3}{2}} = MLT^{-2} \div M^{\frac{1}{2}}L^{\frac{3}{2}} = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}.$$

Dimensions of Magnetic Potential. Magnetic potential is based on the following facts as regards the determination of its dimensions. The product of magnetic intensity by quantity gives energy, just as the product of a volt by a coulomb does in practical units. If the quantity is a unit it is evident that the energy will vary with the potential. In other words the energy when the quantity is of unit value will be numerically equal to the potential. This is a statement widely different from the one so often used, that for unit quantity the potential is "equal to" or "is work or energy." The equality is only numerical and only exists for unit value of quantity. This gives

$$\begin{aligned} e' &= ML^2T^{-2} \text{ (energy)} \div M^{\frac{1}{2}}L^{\frac{3}{2}} \text{ (quantity)} \\ &= M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2} \text{ (magnetic potential).} \end{aligned}$$

Dimensions of Magnetic Power. Magnetic power is the product of magnetic density by thickness of shell, or

$$\phi = M^{\frac{1}{2}}L^{-\frac{3}{2}} \text{ (density)} \times L = M^{\frac{1}{2}}L^{-\frac{1}{2}}.$$

PROBLEMS.

Express 17 amperes in the E.S. system. *Ans.* 51×10^9 E.S. units.

Express 39 volts in the E.S. system.

Ans. 0.13 E.S. unit, or 13×10^{-2} E.S. units.

Express 29 ohms in the E.S. system.

Ans. $3\frac{2}{9} \times 10^{-11}$ E.S. unit, or 322×10^{-13} E.S. units.

Express Ohm's law $\dfrac{1 \text{ volt}}{1 \text{ ohm}} = 1$ ampere in the E.S. system.

Ans. $\frac{1}{3} \times 10^{-2} \div \frac{1}{9} \times 10^{-11} = 3 \times 10^{9}$ E.S. units.

If 13×10^{-2} E.S. unit of potential acts upon a resistance of 322×10^{-12} E.S. units, what will the current be in E.S. absolute, in E.M. absolute, and in practical units? *Ans.* $4{,}037 \times 10^{6}$ E.S. absolute.
0.1346 E.M. absolute.
1.346 ampere.

Express 15 coulombs in E.S. and E.M. absolute units.

Ans. 45×10^{9} E.S. units.
1.5 E.M. units.

What is the ratio of the E.S. to E.M. units of quantity from the above? *Ans.* E.S. = E.M. $\times 3 \times 10^{10}$.

Give the calculation for the watt in E.S. units.

Ans. $3 \times 10^{9} \times \frac{1}{3} \times 10^{-2} = 10^{7}$ E.S. units.

Give the same calculation for E.M. units.

Ans. $10^{8} \times 10^{-1} = 10^{7}$ E.M. units.

How many volts are there in 390 E.S. units of potential?

Ans. 1.30 volts.

How many amperes are there in 277 E.M. units of current?

Ans. 27.7 amperes.

CHAPTER X.

THERMO-ELECTRICITY.

Emissivity. — Heating of a Conductor by a Current. — Cross Section of a Conductor to be Heated to a given Degree by a Given Current.— Thermo-electric Couple. — Electro-motive Force of a Thermo-electric Couple. — Neutral Temperature. — Temperature and Electro-motive Force of Thermo-electric Couple. — Thermo-electric Tables. — Peltier Effect. — Absolute Temperature. — Thomson Effect.

Emissivity. If a body is heated above the temperature of its surroundings it parts with its heat. The process is termed emissivity. Unit emissivity is the quantity of heat lost per second per square centimeter of surface per degree of difference between its temperature and that of its surroundings.

It is measured in ergs or calories or other heat unit. The C.G.S. unit of heat is the erg. The practical unit is the therm, calorie, or gram-degree, which is the heat required to raise one gram of water one degree C.

Heating of a Conductor by a Current. A number of formulas for calculation of the temperature imparted to a conductor by a current passing through it have been proposed. From the nature of the case these formulas can only be approximate.

By radiation and convection about 0.00,025 calorie is lost per second by an unpolished metallic surface of one square centimeter in the air for each degree C. that it is heated above the air. If therefore the calories expended on a conductor by the passage of a current are determined, and if the wire is assumed to have attained its fixed temperature, these calories will be equal to those lost by radiation and convection. The quotient of these calories divided by the area of the surface of the conductor in square centimeters will give the calories per square centimeter which are expended on the conductor and which

are in turn lost by it. The quotient of these calories divided by 0.00,025 is the temperature in degrees C. which the wire will attain. The result is only approximate.

Example. A current of 10 amperes passes through a wire one centimeter in circumference. The wire has a resistance of 0.3 ohm per 100 meters. How many degrees will its temperature be increased?

Solution. The watts per second due to the passage of the current are given by the expression $I^2R = 100 \times 0.3 = 30$. The surface area of 100 meters of the wire are 10,000 square centimeters. The calories expended on 100 meters are $30 \times 0.24 = 7.2$. The calories per square centimeter are the quotient of 7.2 by the area of the wire, or $7.2 \times 10^{-4} = 0.00{,}072$ calorie. Dividing 0.00,072 by 0.00,025 gives 2.88° C., the degrees C. above the temperature of the air to which the wire would be heated by such a current.

A formula for the rise in temperature in degrees C. of a bare copper wire with a current passing through it in still air is given below. The square of the amperes is divided by the cube of the diameter in mils, and the quotient is multiplied by 50,000. The product is the increase in temperature.

$$\text{Temperature C.} = \frac{I^2}{d^3} \times 50{,}000.$$

Example. A copper wire 125 mils in diameter is conducting a current of 10 amperes. How much will it be heated above the temperature of the surrounding air, the wire being of bare copper?

Solution. Substituting in the formula the above values gives

$$\text{Temperature C.} = \frac{10^2}{125^3} \times 50{,}000 = 2.5^\circ \text{ C.}$$

In the above example the thickness of the wire is the same as in the preceding example. The results are reasonably close for approximate methods.

Cross Section of a Conductor to be Heated to a Given Degree by a Given Current. The diameter of a wire of a material of known specific resistance which a stated current will heat to a stated degree of heat above that of the surrounding air can be calculated approximately by the application of the emissivity factor.

The following discussion leads to results in centimeters.

Specific resistance is given in microhms in the tables. The resistance in ohms of a wire of diameter d and whose cross-sectional area is therefore $\pi d^2/4$ is given in the equation

$$R = \text{sp. res.} \times 10^{-6} \div \frac{\pi d^2}{4} = \text{sp. res.} \times 10^{-6} \times \frac{4}{\pi d^2}. \qquad (1)$$

The heat expended on a conductor in the passage of a current is I^2R watt-seconds or joules per second. Substituting for R its value from (1) gives

$$\text{Heat} = I^2 \times \text{sp. res.} \times 10^{-6} \times \frac{4}{\pi d^2} \text{ watt-seconds per second.} \qquad (2)$$

A calorie is equal to 4.16 watt-seconds. If the second member of (2) is divided by 4.16 it becomes

$$\text{Heat} = \frac{I^2 \times \text{sp. res.} \times 10^{-6} \times 4}{4.16 \times \pi d^2} \text{ calories per second.} \qquad (3)$$

The surface area of one centimeter of the wire is πd. If the second member of (3) is divided by $\pi\, d$, the result will be the heat per square centimeter of surface area. Accepting 4,000 as the emissivity factor, the heat emitted per square centimeter at the temperature t is $t/4{,}000$. If the wire attains a constant temperature, the heat expended and that emitted are the same. Therefore if the second member of (3) is divided by πd it can be equated with $t/4{,}000$, giving

$$\frac{t}{4{,}000} = \frac{I^2 \times \text{sp. res.} \times 10^{-6} \times 4}{4.16 \times \pi^2 d^3}. \qquad (4)$$

Multiplying (4) throughout by $\dfrac{d^3 \times 4{,}000}{t}$ gives

$$d^3 = \frac{I^2 \times \text{sp. res.} \times 10^{-6} \times 4 \times 4{,}000}{t \times \pi^2 \times 4.16}$$

$$= \frac{I^2 \times \text{sp. res.} \times 0.00039}{t} \qquad (5)$$

and $$d = \sqrt[3]{\frac{I^2 \times \text{sp. res.} \times 0.00039}{t}}. \qquad (6)$$

To calculate the diameter of a cylindrical conductor which will attain a certain temperature in the air in passing a certain current, multiply the square of the current by the specific resistance and by the factor 0.00039 and divide the result by the temperature. The cube root of the product will be the diameter of the conductor in centimeters.

Example. Calculate the diameter of a lead wire to melt with a current of 7.2 amperes. The melting point of lead is taken as 335° C., and its specific resistance is 19.85 (microhms per cubic centimeter).

Solution. By formulas (5) and (6) the cube of the diameter is obtained by substitution, giving

$$d^3 = \frac{51.84 \times 19.85 \times 0.00039}{335} = 0.0019,$$

$$d = \sqrt[3]{0.00119} = 0.1058 \text{ cm.}$$

Recurring to equation (4), the term $4/\pi^2 d^3$ is the reciprocal of the product of the surface area of the unit length of a conductor by the cross-sectional area of the same. This product is $\pi d^2/4 \times \pi d$, and its reciprocal is $4/\pi^2 d^3$, as it appears in the second member of equation (4). This equation is restricted to a circular conductor. By substituting for $\pi d^2/4$ and for πd the proper values the equation may be made to apply to conductors of other cross sections.

Suppose a strip is to be cut from a sheet of metal of thickness a, and it is to be calculated how wide the strip shall be to attain a given temperature with a given current. Call the width of the strip na, the factor n being the unknown quantity to be determined. The cross-sectional area of the strip will be $na \times a = na^2$. The surface area of the unitary length will be $2\,na + 2a = 2a(n+1)$. The product of these two quantities is $2a^3(n^2+n)$. As this is the product of the surface area of the unit length of the conductor by its cross-sectional area, it is obvious that it can be substituted for $\frac{\pi^2 d^3}{4}$ in equation (4). This substitution is performed by division by it in place of division by $\frac{\pi^2 d^3}{4}$, giving

$$\frac{t}{4{,}000} = \frac{I^2 \times \text{sp. res.} \times 10^{-6}}{4.16 \times 2\,a^3(n^2+n)}. \tag{7}$$

Multiplying by $\frac{(n^2+n) \times 4{,}000}{t}$ gives

$$n^2 + n = \frac{I^2 \times \text{sp. res.} \times 10^{-6} \times 4{,}000}{4.16 \times t \times 2\,a^3}$$

$$= \frac{I^2 \times \text{sp. res.} \times 0.0{,}004{,}808}{t \times a^3}. \tag{8}$$

The value of n is obtained from this equation by substituting the constants given in the problem. The width of the strip is the product of its thickness by n, or na.

Example. A strip of lead to be melted by a current of 7.2 amperes is to be cut from a sheet 0.01 cm. thick. The specific resistance of lead is taken as 19.85 and its melting point as 335° C. Calculate the width of the strip.

Solution. Substituting these values in equation (8) gives

$$n^2 + n = \frac{(7.2)^2 \times 19.85 \times 0.0{,}004{,}808}{335 \times (0.01)^3} = 1{,}476.88.$$

Solving this we find

$$n = 38, \text{ approximately.}$$

The width of the strip is the product of the thickness by this factor. It is therefore

$$38 \times 0.01 = 0.38 \text{ cm.}$$

Thermo-electric Couple. If two conductors of different materials, such as iron and copper, are joined at both ends and separated at the middle so as to form a species of ring, and if the junctions are maintained at different temperatures with proper relation to what is called the neutral temperature a current of electricity will be produced, continuing as long as the difference of temperature is maintained. It is sufficient if one pair of ends is in actual contact, the pieces then separating and having their other ends connected by a wire. The arrangement described constitutes a thermo-electric couple. The intensity of the current produced depends upon the e.m.f. and upon the resistance of the circuit.

Electro-motive Force of a Thermo-electric Couple. The e.m.f. produced by a thermo-electric couple depends upon the difference of temperature of the ends referred to the neutral temperature of the particular couple. If the junction of a couple is heated above the neutral temperature a certain number of degrees and if the other ends are the same number of degrees below the neutral temperature there will be no e.m.f. produced. Different couples have widely different neutral temperatures. For each couple there is a fixed neutral temperature.

Neutral Temperature. When the mean temperature of the ends of the couple is below the neutral temperature the current flows in one direction; when the mean temperature is above the neutral temperature the current flows in the other direction.

Example. The neutral temperature of a thermo-electric couple of two particular metals is 275° C. One end is heated to a temperature of 300° C., the other end to a temperature of 250° C. What will the result be?

Solution. The average temperature is $(250° + 300°) \div 2 = 275°$. Thus the mean or average temperature of the ends is the neutral temperature of the couple. No potential difference will be maintained and consequently no current will pass. The result as far as the electric state of things is concerned will be zero.

Example. The junction of a couple is heated to a temperature of 675° F., the other ends are kept at a temperature of 55° F. and no current nor difference of e.m.f. can be produced. What is the neutral point of the combination?

Solution. The average temperature is $(675° + 55°) \div 2 = 365°$. As no current is produced at this mean temperature of the ends, or, what is the same in effect, as no e.m.f. is produced at this mean temperature, the neutral temperature of the couple is 365° F.

Temperature and Electro-motive Force of Thermo-electric Couple. The e.m.f. produced by a thermo-electric couple depends on the difference of temperature maintained between the opposite ends of the couple. The e.m.f. produced by various combinations of elements of thermo-electric couples, such as an iron-copper couple, a bismuth-antimony couple, and so on, is different for different couples.

To increase the e.m.f. of a thermo-electric battery the couples must be arranged in series. The laws given for the arrangement of battery cells apply in general to thermo-electric couples.

Thermo-electric Tables. A thermo-electric table is a table of constants by which the e.m.f. of different couples can be calculated. In some such tables no allowance is made for the neutral temperature. It is obvious that such a table is only

accurate for one mean temperature. This is not a serious defect, because the mean temperature of a thermo-electric battery is not subject to a wide variation.

THERMO-ELECTRIC TABLE FOR A NEUTRAL TEMPERATURE OF ABOUT 20° C.

(Jenkin's "Electricity and Magnetism" compiled from Matthiessen's experiments reduced to C.G.S. units.)

Bismuth, pressed commercial wire	− 9,700
Bismuth, pure pressed wire	− 8,900
Bismuth, crystal, axial	− 6,500
Bismuth, crystal, equatorial	− 4,500
Cobalt	− 2,200
German silver	− 1,175
Mercury	− 41.8
Lead	0
Tin	+ 10
Copper, commercial	+ 10
Platinum	+ 90
Gold	+ 120
Antimony, pressed wire	+ 280
Silver, pure hard	+ 300
Zinc, pure pressed	+ 370
Copper, electrolytic	+ 380
Antimony, pressed commercial wire	+ 600
Arsenic	+ 1,356
Iron, pianoforte wire	+ 1,750
Antimony, crystal, axial	+ 2,260
Antimony, crystal, equatorial	+ 2,640
Phosphorus, red	+ 2,970
Tellurium	+ 50,200
Selenium	+ 80,700

To calculate the e.m.f. of a combination subtract algebraically the number opposite one element of the couple from the number opposite the other element. The result is the e.m.f. in C.G.S. units corresponding to a difference of temperature of 1° C. between the ends of the couple. The polarity of the e.m.f. is such as to produce a current from the lower to the upper metal through the hotter end or junction.

Example. Calculate the e.m.f. between lead and cobalt at a mean temperature of 20° C. with a temperature difference of 1° C. between the junctions or ends.

Solution. The numbers opposite the metals in the table are 0 and $-2,200$. Subtracting gives 2,200 C.G.S. units of e.m.f. As 10^8 C.G.S. units = 1 volt, this is $2,200 \div 10^8 = 2,200 \times 10^{-8} = 0.000,022$ volt.

Example. Assume that the elements in the above example are joined in contact at one end, while a wire connects their free ends, and that the heat is applied to the junction. In which direction will the current go?

Solution. The heated junction is the hottest. Cobalt is the lower metal in the table. Therefore the polarity of the e.m.f. is such as to produce a current from the cobalt bar through the hotter junction to the lead.

Example. If the difference of temperature in the above case were 7° C., what would the voltage be?

Solution. $0.000,022 \times 7 = 0.000,154$ volt.

Example. A thermo-electric battery of German silver and iron elements at a mean temperature of 20° C. with a temperature difference between the ends of the couples of 121° C. and with the couples arranged in series gives an e.m.f. of 1.33 volts. Calculate the number of couples in the series.

Solution. Iron, $+ 1,756$, and German silver, $-1,175$, are to be subtracted algebraically. Changing the sign of German silver and adding we obtain $1,756 + 1,175 = 2,931$. This is the e.m.f. of a single couple in C.G.S. units at 1° C. difference of temperature between the ends. At 121° C. the difference is $2,931 \times 121 = 354,651$ C.G.S. units. 1.33 volts (the e.m.f. of the battery in volts) $= 1.33 \times 10^8 = 133,000,000+$ C.G.S. units. Dividing the e.m.f. of the battery in C.G.S. units by the e.m.f. of a single couple in the same units gives the number of couples. $133,000,000 \div 354,651 = 375$, which is the number of couples required to produce 1.33 volts at the difference of temperature specified in the problem.

The e.m.f.'s of the above calculation could have been expressed

in volts and the operation could have been done in these units. It is merely a question of decimal place. Thus:

354,651 C.G.S. units = 0.003,546,510 volt and
1.33 ÷ 0.003,546,510 = 375 couples, as before.

In these operations it is immaterial which number has its sign changed, becoming thereby the subtrahend. It is only necessary to change the sign of one of the numbers and to add the two algebraically. The position in the table determines the polarity.

The table given below takes in a considerable range of mean temperature and is a more satisfactory one to work with than is the last. It is based upon the work of Professor Tait (Trans. Royal Soc., Edinburgh, Vol. XXVII, 1873). The table is taken from Everett's "C.G.S. System of Units."

THERMO-ELECTRIC HEIGHTS AT t° C IN C.G.S. UNITS.

Iron	+1,734	− 4.87 t
Steel	+1,139	− 3.28 t
Alloy, platinum, 95; iridium, 5	+ 622	− 0.53 t
Alloy, platinum. 90; iridium, 10	+ 596	− 1.34 t
Alloy, platinum, 85; iridium, 15	+ 709	− 0.63 t
Soft platinum	− 61	− 1.10 t
Alloy, platinum and nickel	+ 544	− 1.10 t
Hard platinum	+ 260	− 0.75 t
Magnesium	+ 244	− 0.95 t
Cadmium	+ 266	+ 4.29 t
Zinc	+ 234	+ 2.40 t
Silver	+ 214	+ 1.50 t
Gold	+ 283	+ 1.02 t
Copper	+ 136	+ 0.95 t
Lead	0	
Tin	− 43	+ 0.55 t
Aluminum	− 77	+ 0.39 t
Palladium	− 625	− 3.59 t
German silver	−1,207	− 5.12 t
Nickel to 175° C	−2,204	− 5.12 t
Nickel 250° to 310° C	−8,449	+24.1 t
Nickel from 340° C	− 307	− 5.12 t

To use the table the mean temperature of the couple in degrees C. is substituted for t.

Example. Calculate the e.m.f. of an iron-German silver couple, the temperatures of the junctions being 0° C. and 100° C.

Solution. The mean temperature is the average temperature of the ends, or $(100 + 0) \div 2 = 50°$ C. This is to be substituted for t of the table, to obtain the values of the elements of the couple, thus:

Value of iron	$1,734 - 4.87\,t$
Value of German silver	$-1,027 - 5.12\,t$
Algebraic difference	$2,941 + 0.25\,t$

Substituting for t its value, 50, in the third expression, we have e.m.f. (at 1° C.) of couple $= 2,941 + (0.25 \times 50) = 2,953.5$ C.G.S. units. The difference of temperature, by the conditions of the problem, is 100 degrees; we therefore have to multiply the e.m.f. for 1° C., as calculated above, by 100 to obtain the e.m.f. for 100 degrees difference, or

e.m.f. (at 100° C. difference of temperature) $= 2,953.5 \times 100 = 295,350$ C.G.S. units. As 10^8 C.G.S. units are equal to 1 volt, we have

$$295,350 \div 10^8 = 295,350 \times 10^{-8} = 0.0,0295 \text{ volt.}$$

The value of t could have been substituted in the expressions for the values of the elements of the couple, thus:

Value of iron ...	$1,734 - 4.87\,t =$	$1,734 - (4.87 \times 50) =$	$1,490.5$
Value of German silver.........	$-1,207 - 5.12\,t =$	$-1,207 - (5.12 \times 50) =$	$-1,463$
Algebraic difference			$2,953.5$

giving the same result as before, the value of the couple for 1° C.

Example. Calculate the neutral point of the above couple.

Solution. The neutral point is the value of t when there is no e.m.f., or when $2,491 + 0.25\,t = 0$, whence $t = -9,964°$ C.

There is therefore no neutral point for this couple, the calculated one being below the absolute zero.

Example. Calculate the neutral point of iron and copper.

Solution. Proceeding as before we find from the table:

Value of iron	$1{,}734 - 4.87\,t$
Value of copper	$136 + 0.95\,t$
Algebraic difference	$1{,}598 - 5.82\,t$

Making the algebraic difference zero gives to t a value which is that of the neutral point. Thus:

$$1{,}598 - 5.82\,t = 0, \text{ whence } t = 275^\circ \text{ C.}$$

Peltier Effect. If a current passes through a circuit of different metals there is a production of heat or a reduction of heat at the junctions of different metals independent of the ordinary heating effect of the current. This is called the Peltier effect. The value of the Peltier effect is expressed in ergs per second per C.G.S. unit current (10 amperes).

Absolute Temperature. To determine the Peltier effect multiply the difference of thermo-electric heights at the junction in question by the absolute temperature of the junction, all in degrees C. The absolute temperature is obtained by adding to the temperature C. the number 273, which is the absolute temperature of the centigrade zero.

Example. What is the absolute temperature of boiling water?

Solution. The temperature C. of boiling water is 100°. To obtain the absolute temperature add 273, or

$$100 + 273 = 373^\circ \text{ absolute temperature C.}$$

Example. Assume a C.G.S. unit current to pass through a junction of iron and copper maintained at a temperature of 100° C. What is the value of the Peltier effect?

Solution. The thermo-electric height of iron at 100° C. is $1,734 - (4.87 \times 100) = 1,247$. That of copper is $135 + (0.95 \times 100) = 230$. The difference of thermo-electric heights is $1,247 - 230 = 1,017$. The absolute temperature at the junction is $100 + 273 = 373$. Then following the rule we have

$1,017 \times 373 = 379,341$ ergs per second for a C.G.S. unit current.

If the current flows from iron to copper through the heated junction it will be a current from the upper to the lower element. This produces a heating effect. If the current is from the lower to the upper element, in this case from copper to iron through the hot junction, the effect will be to cool the junction. In other words, if the direction of the current is such as to supplement the thermo-electric current, it will cool the junction, otherwise the reverse will hold.

Thomson Effect. If a conductor of one material throughout is heated in places so that some parts are hotter than others, it will show thermo-electric action, except in the case of lead. The action is called the Thomson effect.

Referring to the table (page 135), metals with a minus sign prefixed to their temperature coefficients, in the second column of figures, are affected as follows: An electric current passing from a hotter to a cooler portion reduces the temperature of the cooler part. Passing from a cooler to a hotter portion it heats the conductor in the hotter part. This applies to iron and all other metals with negative signs in the table.

Metals with a positive sign prefixed are affected in the reverse way. A current passing from hot to cool heats the cool portion. Passing from cool to hot it lowers the temperature of the hot portion. Copper is one of the metals subject to this action.

The general law is that in the case of metals with a negative

sign an electric current tends to increase any local differences of temperature already existing. In the case of metals with positive signs an electric current tends to equalize temperature differences.

If portions of a conductor differ in temperature, the product of that difference by the thermo-electric temperature coefficient of the metal as given in the table gives the thermo-electric difference between the portions in question. The temperature difference is to be in degrees C., and the result will be in C.G.S. units.

Example. The ends of an iron wire differ by 100° C. in temperature. Calculate the thermo-electric difference between the ends.

Solution. From the table the thermo-electric coefficient of iron is found to be -4.87. Then, as the sign has no effect upon the multiplication except as indicating the polarity of the e.m.f., we have

$100 \times 4.87 = 487$ C.G.S. units of e.m.f., or 0.000,004,87 volt.

As the coefficient has a negative sign, the current tends to increase any existing differences of temperature.

The value of the Thomson effect in ergs per second per C.G.S. unit current (10 amperes) is found by multiplying the thermo-electric difference, as just calculated in the case of iron for instance, by the sum of one-half the temperature difference in degrees C. and 273. This reduces the temperature to the absolute scale C.

Example. Calculate the Thomson effect for the iron wire of the last example.

Solution. The temperature difference is 100° C.; half of this is 50. Following the rule we have

$487 \times (273 + 50) = 157{,}301$ ergs per second for a current of 10 amperes (1 C.G.S. unit).

If the current goes from cold iron to hot iron, this number of

ergs of electric energy are converted into heat energy; if the current goes the other way, the same number of ergs of heat energy are converted into electric energy.

Example. Let a copper wire be unequally heated 100° C. as in the last example. Calculate the Thomson effect.

Solution. The thermo-electric effect will be $0.95 \times 100 = 95$ C.G.S. units. For the ergs per second we have

$95 \times (273 + 50) = 30{,}685$ ergs per second for a 10-ampere current.

Example. Make the same calculation for lead.

Solution. As the temperature coefficient for lead is 0, there is no Thomson effect for lead.

PROBLEMS.

Calculate the e.m.f. between iron and lead per degree C. at 0° C. *Ans.* $1{,}734 \times 10^{-8}$ volt.

Calculate the e.m.f. of iron and copper, the junctions being kept at 0° C. and 100° C. *Ans.* $130{,}700 \times 10^{-8}$ volt.

A thermo-electric battery of 33 bismuth (− 9,700) and antimony (2,250) couples shows 1.25 volts at the mean temperature of 19° C. What is the difference of temperature between the two faces? *Ans.* 317° C.

A current of 11 amperes is passed through a wire of 0.7 cm. circumference and of 0.6 ohm resistance per 100 meters. How much will its temperature be increased? *Ans.* 10° C.

What will be the rise in temperature in a copper wire of 110 mils diameter carrying a current of 20 amperes? *Ans.* 15° C.

Calculate the diameter of a lead wire to melt with a current of 3.1 amperes. *Ans.* 0.0606 cm.

Taking the melting point of copper as 1,127° C. and its specific resistance as 1.652, calculate the thickness of a wire that will melt at 21 amperes current. *Ans.* 0.0632 cm.

A copper strip is 0.02 cm. thick. How wide should it be to melt with an 18-ampere current? *Ans.* 0.097+ cm.

How wide should the same copper strip be for ten times as great a current? *Ans.* 1.0586 cm. (more than ten times the first width).

Calculate the e.m.f. of bismuth (– 9,700) and platinum per degree C. at 20° mean temperature C. *Ans.* 0.000,097,9 volt.

Make the same calculation for German silver and copper.

Ans. 0.000,015,55 volt.

How many couples of the above would be required for a volt at 100° C. difference of temperature? *Ans.* 608 couples.

Calculate by table on page 190 the e.m.f. per degree C. difference, of German silver and copper at 20° C. mean temperature.

Ans. 0.000,014,64 volt.

What is the e.m.f. per degree C. for a mean temperature of 315° C.?

Ans. 0.000,032,55 volt.

At 600° C. difference of temperature what would be the number of couples of the combination of the last problem to a volt?

Ans. 53 couples.

What is the neutral point of the above combination? *Ans.* – 221° C.

What is the neutral point of iron and zinc? *Ans.* 206° C.

What is the neutral point of aluminum and palladium?

Ans. – 138° C.

Calculate the e.m.f. per degree C. of aluminum and nickel at a mean temperature of 300° C.? *Ans.* 0.000,012,69 volt.

What would the voltage be at 600° C. difference of temperature, at above mean temperature, and how many couples would be required for a volt? *Ans.* 0.007,614 volt; 132 couples.

Calculate the Thomson effect in a copper wire, with a difference of 200° C. in temperature of different sections of its length.

Ans. 0.000,001,9 volt; 70,870 ergs per second.

Make the same calculation for German silver with 75° C. difference of temperature. *Ans.* 0.000,003,84 volt; 119,232 ergs per second.

Make the same calculation for silver with 98° C. difference in temperature. *Ans.* 0.000,001,47 volt; 47,334 ergs per sec.

What is the Peltier effect between German silver and iron at 100° C.

Ans. 1,106,318 ergs per second for a C.G.S. unit current.

Calculate the Peltier effect for zinc and tin at 190° C.

Ans. 290,995 ergs per second for a C.G.S. unit current.

CHAPTER XI.

ELECTRO-CHEMISTRY.

Chemical Composition. — Chemical Saturation. — Chemical Equivalents and Atomic Weights. — Electric Decomposition or Electrolysis. — Relation of Chemical Equivalents to Electrolysis. — Electro-chemical Equivalents. — Thermo-electro Chemistry. — Calculation of Electro-motive Force of a Voltaic Couple. — Problems.

Chemical Composition. The atomic weights of the elements indicate the proportions in which the elements combine, subject to Avogadro's law. Thus the atomic weight of hydrogen being 1 and that of oxygen being 16, it follows that in their combination with each other the ratio of 1 : 16 or some multiple thereof must obtain. The water molecule contains two atoms of hydrogen and one atom of oxygen. The ratio follows from the atomic weights; it is 2 : 16. The atomic weight of the hydrogen is multiplied by 2. This example illustrates a fundamental law of chemistry, the law of multiple proportions.

The determination of the atomic weights is one of the most difficult problems of the analytic branch of chemistry, and atomic weights are subject to constant revision as new determinations are made.

Example. The formula of sulphuric acid is H_2SO_4. Calculate the proportions of the constituent elements.

Solution. Multiplying the atomic weight of each element by the number of the atoms of it which are in the molecule we have

$$H = 2 \times 1 = 2 \cdot S = 1 \times 32 = 32 \cdot = 4 \times 16 = 64.$$

This gives the ratio

$$H : S : O : : 2 : 32 : 64.$$

Example. The formula of sodium sulphate is Na_2SO_4. Calculate the proportions of the constituent elements.

Solution. Sodium : sulphur : oxygen : : 46 : 32 : 64.

Example. Calculate the proportions of the constituent elements of silver nitrate, $AgNO_3$ (N = 14).

Solution. Silver : nitrogen : oxygen : : 107.1 : 14 : 48.

Chemical Saturation. When two elements combine so as to satisfy their chemical affinities they are said to saturate each other. If a single atom of one element saturates a single atom of another element, the two are said to have the same valency. Thus the formula of copper oxide is CuO. It is completely saturated; therefore both copper and oxygen are of the same valency, or atomicity, as it is also called.

Valency. If other ratios than the unitary obtain in a simple saturated molecule, it indicates that the valencies of the constitutent elements differ one from the other. The formula of the water molecule is H_2O; therefore oxygen has a valency twice as great as that of hydrogen.

Elements of a valency of one are called monads; those of a valency of two, dyads; of three, triads; of four, tetrads; of five, pentads; of six, hexads; of seven, heptads; of eight, octads.

Example. The valency of hydrogen is 1. What is the valency of oxygen, the water molecule being a saturated one and its formula being H_2O?

Solution. As it requires two atoms of the monad hydrogen to saturate the one atom of oxygen, oxygen must be a dyad.

Example. The formula of sulphur trioxide is SO_3. What is the valency of sulphur ?

Solution. As one atom of sulphur saturates three atoms of the dyad oxygen, the valency of sulphur is $3 \times 2 = 6$; sulphur is a hexad.

The statements given above are rather illustrative than exhaustive, as other considerations apply in many cases in chemistry. It is also to be noted that in practical work the

less important decimals are omitted from the atomic weights in calculations.

Chemical Equivalents and Atomic Weights. The chemical equivalent of an element is the quotient of its atomic weight divided by its valency. It expresses the simplest ratio of the constituent elements of any simple saturated molecule of two elements, a binary molecule as it is termed.

The atomic weight of oxygen is 16; it is a dyad; its chemical equivalent is therefore $16 \div 2 = 8$. The atomic weight of hydrogen is 1; it is a monad; therefore its chemical equivalent is 1. The molecule of water has the formula H_2O. Substituting for the constituent symbols their chemical equivalents gives 1 : 8 as the ratio of the oxygen and hydrogen in water.

Example. Calculate the chemical equivalents of sulphur.

Solution. Sulphur in some compounds is a hexad. Its atomic weight is 32; its chemical equivalent is $32 \div 6 = 5.3$. In other compounds, in SO_2 for example, it is a tetrad. The chemical equivalent of tetrad sulphur is $32 \div 4 = 8$. Sometimes, as in H_2S, it is a dyad, when its chemical equivalent is $32 \div 2 = 16$.

Electric Decomposition or Electrolysis. When a current of electricity is passed through a compound so as to electrolyze it, a definite amount of the compound is decomposed by a definite amount of electricity. For various reasons some compounds cannot be electrolyzed. The statement of the necessary requirements for electrolysis is outside the field of this book. Assuming the possibility of the separation of any element from its compounds by electrolysis, the quantity of an element actually or potentially separable by a definite quantity of electricity will be referred to as the weight or quantity "corresponding to" the quantity of electricity in question. In the same way the quantities of electricity "corresponding to" given weights of the elements and of their compounds will be referred to.

	Atomic Weight.	Valency.	Chemical Equivalent.	Electro-chemical Equivalent for C.G.S. Unit.	Reciprocal.
Chlorine	35.18	1	35.18	.003672	272.3
Chromium	51.7	6	8.62	.000900	1,111.4
Copper (cuprous)	63.1	1	63.1	.006586	151.83
Copper (cupric)	....	2	31.55	.003293	303.66
Gold	195.7	3	65.23	.006809	146.87
Hydrogen	1	1	1	.00010438	9,580.4
Iron (ferrous)	55.5	2	27.75	.002897	345.24
Iron (ferric)	....	3	18.50	.001931	517.86
Lead	205.35	2	102.67	.010717	93.31
Mercury (—ous)	198.5	1	198.5	.020719	48.26
Mercury (—ic)	....	2	99.25	.010360	96.53
Nickel	58.3	2	29.15	.003043	328.66
Oxygen	15.88	2	7.94	.000829	1,206.6
Platinum	193.3	6	32.22	.003363	297.34
Silver	107.11	1	107.11	.01118	89.45
Sodium	22.88	1	22.88	.002388	418.72
Sulphur	31.82	2	15.91	.001661	602.16
Sulphur	....	4	7.95	.000831	1,204.32
Sulphur	....	6	5.30	.000553	1,806.48
Tin (stannous)	118.1	2	59.05	.006164	162.24
Tin (stannic)	....	4	29.52	.003082	324.48
Zinc	64.9	2	32.45	.003387	295.24

It is not necessary for ordinary work to use more than one decimal, and in many cases none need be expressed. Thus 107 may often be used as the atomic weight of silver, and 16 is almost always used as the atomic weight of oxygen.

Relation of Chemical Equivalents to Electrolysis. Chemical equivalents are the relative weights of elements corresponding to any definite quantity of electricity.

In some molecules atoms of the same element partly saturate each other. Chemical equivalents do not apply to such. In the molecule C_2H_6 the tetrad carbon elements partly saturate each other, so that the carbon is virtually a triad. In the molecule C_6H_{14} it has a virtual valency of $2\frac{1}{3}$.

The quantities of elements which a definite quantity of electricity will precipitate are proportional to their chemical equivalents.

Example. A certain quantity of electricity would precipitate 119 grams of silver. What weight of copper would the same quantity precipitate?

Solution. The chemical equivalents of silver and of copper are respectively 107 and 31.6. We then have the proportion

$$107.1 : 31.6 :: 119 : x = 35.1 \text{ grams.}$$

Example. A current of electricity is passed through two solutions, one a solution of silver, the other a solution of an unknown salt. From the first solution 129 grams of silver are precipitated; from the other solution 35.1 grams of an unknown metal are precipitated. What was the metal?

Solution. The ratio of the metals precipitated is the ratio of their chemical equivalents. This gives the proportion

$$129 : 35.1 :: 107.1 \text{ (the chemical equivalent of silver)} : x$$
$$\text{(the chemical equivalent of the unknown metal)} = 29.1.$$

By referring to the table this is seen to be the chemical equivalent of nickel, which therefore is the metal of the unknown solution.

Electro-chemical Equivalents. The electro-chemical equivalent of an element is the weight in grams corresponding to,

or which would be precipitated by, one C.G.S. unit of electricity. A table could be made out for any desired unit.

The electro-chemical equivalents of the elements are proportional to their chemical equivalents.

Example. The electro-chemical equivalent of hydrogen is 0.000,104,38. What is the electro-chemical equivalent of sodium?

Solution. The chemical equivalent of sodium is 23 and that of hydrogen is 1; therefore its electro-chemical equivalent is the product of $0.000,104,38 \times 23 = 0.002,401$.

Example. The electro-chemical equivalent of silver is 0.01118. Calculate the electro-chemical equivalent of zinc by direct proportion.

Solution. The chemical equivalents of silver and of zinc are respectively 107.1 and 32.45, retaining the decimals. As the electro-chemical equivalents are proportional to the chemical equivalents the proportion follows:

$107.1 : 32.45 :: 0.01118 : x = 0.003,387$, which is the electro-chemical equivalent of zinc.

Example. Calculate the electro-chemical equivalent of a dyad element whose atomic weight is 118.1.

Solution. The chemical equivalent is the quotient of the atomic weight divided by the valency, or $118.1 \div 2 = 59.05$. The electro-chemical equivalent is the product of the electro-chemical equivalent of hydrogen, 0.000,104,38, by this number.

$0.000,104,38 \times 59 = 0.006,164$, which is the electro-chemical equivalent of stannous tin.

The reciprocal of the electro-chemical equivalent is the number of units of electricity required to precipitate one gram of the substance or element in question. Thus to precipitate one gram of hydrogen $1 \div 0.000,104,38 = 9,580.4$ C.G.S. units of electricity are required.

To reduce this to coulombs multiply by 10. To precipitate one gram of hydrogen 9,580.4 coulombs are required.

To obtain the electro-chemical equivalent of any element the electro-chemical equivalent of hydrogen may be multiplied by the chemical equivalent of the element. To obtain the reciprocal of any element, divide the reciprocal of hydrogen, 9,580.4, by the chemical equivalent of the element.

Example. Calculate the electro-chemical equivalent and its reciprocal for the metal lead.

Solution. The electro-chemical equivalent of hydrogen is 0.000,104,38. Multiplying this by the chemical equivalent of lead, 102.67, gives 0.010,717, the electro-chemical equivalent of lead. Dividing 9,580.4 by 102.67 gives 93.31, the reciprocal for lead.

Example. In one hour a current of electricity precipitates 3.740 grams of silver. What is the strength of the current?

Solution. There are 3,600 seconds in an hour. If we divide 3.740 by 3,600, the quotient, 0.001,039, is the quantity of silver precipitated in one second. One ampere precipitates 0.001,118 gram of silver in one second. Therefore 0.001,039 ÷ 0.001,118 = 0.93 amperes is the strength of the current.

Example. How many grams of silver are deposited in an hour by one ampere?

Solution. One ampere-hour is equal to 3,600 coulombs. One C.G.S. unit deposits 0.01118 gram, and one coulomb deposits 0.001,118 gram, because one C.G.S. unit is equal to 10 coulombs. 0.001,118 × 3,600 = 4.0248 grams, which is the weight of silver deposited in an hour by one ampere, or by 10 amperes in $\frac{1}{10}$ hour.

Example. For how long must a current of 7 amperes be maintained to precipitate 13 grams of silver?

Solution. From the last column we find that to precipitate one gram of silver 89.45 C.G.S. units are required. These are

equal to 894.5 coulombs. To precipitate 13 grams $894.5 \times 13 = 11{,}628.5$ coulombs are required. A current rate of 7 amperes is 7 coulombs per second, so that $11{,}628.5 \div 7 = 1{,}661.2$ seconds, or 27 minutes 41 seconds would be required.

Thermo-electric Chemistry. Let H be the heat in calories due to the combination of one gram of any element with another. Assume that Q coulombs have acted; then if we denote the electro-chemical equivalent of the element by z, the weight of the element involved in the reaction will be Qz grams, and the heat will be QzH calories. One calorie is equal to 0.424 kilogram-meters of energy. Therefore the kilogram-meters of the reaction will be

$$0.424\ QzH \ldots \qquad (1)$$

The energy of the reaction can be expressed by direct reference to electrical energy. If Q coulombs are involved in a reaction, the energy is equal to $Q \times E$ joules, E denoting the volts of the same reaction. One joule is equal to 1/9.81 kilogram-meters Therefore the kilogram-meters of the reaction will be

$$\frac{QE}{9.81} \ldots \qquad (2)$$

Equating (1) and (2) we have

$$\frac{QE}{9.81} = 0.424\ QzH \ldots \qquad (3)$$

and solving (3) we find

$$E = 4.16\ zH \ldots \qquad (4)$$

Calculation of Electro-motive Force of a Voltaic Couple. This expression gives the value of the e.m.f. of a reaction in terms of the electro-chemical equivalent and of the calories of heat per gram. The latter quantity has been determined for a great many chemical reactions.

Example. What e.m.f. is required to decompose water?

Solution. Take the value 34,450 calories as the heat due to

the oxidation of one gram of hydrogen, producing water. The electro-chemical equivalent of hydrogen is, for coulomb notation, 0.000,010,438. Introducing these values in (4) gives

$$E = 4.16 \times 0.000{,}010{,}438 \times 34{,}450 = 1.495 \text{ volts.}$$

These calculations are not in exact accord with direct experiment, as the heat coefficient is not always exact, and subsidiary reactions may exist which are not taken into account in the calculation.

In the action of a battery, chemical decomposition, operating to reduce the e.m.f., sometimes has a place. Then in calculating the e.m.f. of such a battery the e.m.f. of decomposition must be subtracted from the e.m.f. of combination to get the e.m.f. of the battery.

Example. Calculate the e.m.f. of the Daniell battery.

Solution. In the Daniell couple zinc is dissolved in sulphuric acid. In the solution of one gram of zinc in sulphuric acid 1,670 calories are set free. The electro-chemical equivalent of zinc is, for coulomb notation, 0.000,338,7. Substituting in (4) we have

$$E = 4.16 \times 0.000{,}338{,}7 \times 1{,}670 = 2.353 \text{ volts.}$$

This is the voltage due to the chemical combination of the couple. There is also a decomposition, that of the copper sulphate, from which copper is deposited. In the solution of copper in sulphuric acid 881 calories are set free or are absorbed in its separation, the two actions being strictly reciprocal. This reciprocal relation obtains in all chemical reactions. The electro-chemical equivalent of copper is 0.000,329,3. Substituting in (4) we have

$$E = 4.16 \times 0.000{,}329{,}3 \times 881 = 1.207 \text{ volts.}$$

As copper is separated in this reaction heat is absorbed and the voltage is reduced. The net voltage of the couple is therefore

$$2.353 - 1.207 = 1.146 \text{ volts.}$$

By actual test the voltage of the Daniell couple is found to be 1.079 volts, a discrepancy of 0.067 volts.

In e.m.f. or polarization calculations such as those just illustrated the calculation may be based on any of the constituents of the reaction or on the final product. The result of the calculation will be the same. To prove this we may use the reaction of hydrogen and oxygen in the formation of water.

In the reaction in question 1 part of hydrogen unites with 8 parts of oxygen to form 9 parts of water. The calories due to the reaction are 34,450 for 1 gram of hydrogen, as we have seen. This then is the heat due to the combination of 8 grams of oxygen with hydrogen, or due to the formation of 9 grams of water. Therefore for 1 gram of oxygen the heat is $34,450 \div 8 = 4,306$ calories, and for 1 gram of water it is $34,450 \div 9 = 3,828$ calories. In applying the formula these are the quantities which have to be multiplied by the electro-chemical equivalents of oxygen or water, as the case may be. But the electro-chemical equivalent of oxygen is equal to that of hydrogen multiplied by 8, and that of water is equal to that of hydrogen multiplied by 9. The result of the multiplication of the formula is therefore the same in all the three cases. The electro-chemical equivalent of oxygen is 0.000,082,9; that of water is 0.000,093,3. The formulas for the oxygen and water basis are obtained by substituting the respective factors in (4), giving

For oxygen, $E = 4.16 \times 0.000,083 \times 4,306 = 1.487$ volts,

For water, $E = 4.16 \times 0.000,093,34 \times 3,828 = 1.487$ volts,

which are the same as those obtained on the hydrogen basis.

The quantity of electricity required for depositing the number of grams of an element equal to its chemical equivalent is the same for all elements. It is 9,580.4 C.G.S. units, or 95,804 coulombs. Thus this quantity of electricity will deposit 107.1 grams of silver, 29.15 grams of nickel, 65.23 grams of gold, 29.52 grams of tin, and so on.

The e.m.f. of a couple can be determined from this factor. The energy of a cell is expressible in two ways. It can be expressed in energy units, such as ergs, or in compound electric units, each one the product of a unit of e.m.f. by a unit of quantity. These two must be equal to each other. If the quantity of electricity is known, it is obvious that the quotient of the energy unit divided by the quantity unit will give the e.m.f.

Example. Calculate the e.m.f. of the Daniell couple, using the above factor.

Solution. 9580.4 C.G.S. units of electricity will precipitate 32.45 grams of zinc. The calories of heat due to the solution of 1 gram of zinc we have taken as 1,670. The calories due to 32.45 (the chemical equivalent of zinc) grams of zinc are equal to $32.45 \times 1{,}670 = 54{,}191.5$ calories.

The calories for the corresponding figure for copper are given by the product of 31.55 (the chemical equivalent of copper) by 881 (the calories corresponding to 1 gram of copper). Thus $31.55 \times 881 = 27{,}796$ calories.

The total calories of the couple are equal to $54{,}191.5 - 27{,}796 = 26{,}395.5$ calories. The copper which is deposited involves the reduction of the heat units.

To reduce calories to C.G.S. units or ergs they must be multiplied by 4.2×10^7.

$$26{,}395.5 \times 4.2 \times 10^7 = 11{,}086 \times 10^8 \text{ ergs.}$$

These ergs correspond to a definite quantity of electric energy. Of this quantity one constituent is known. This constituent is 9,580.4 C.G.S. units of quantity. If therefore the ergs are divided by this figure, the quotient will be the other constituent or factor. This other factor is e.m.f. expressed in C.G.S. units.

$$(11{,}086 \times 10^8) \div 9{,}580.4 = 1.16 \times 10^8 \text{ C.G.S. units} = 1.16 \text{ volts.}$$

Example. Calculate the e.m.f. of the decomposition of water.

Solution. The chemical equivalent of hydrogen being 1, the

calories per equivalent are the same as those per gram, namely, 34,450. The operation is then the same as the last, except that the calories per gram of equivalent do not have to be calculated.

$$34{,}450 \times 4.2 \times 10^7 = 14{,}469 \times 10^8 \text{ ergs.}$$

$$(14{,}469 \times 10^8) \div 9{,}580.4 = 1.51 \times 10^8 \text{ C.G.S. units} = 1.51 \text{ volts.}$$

PROBLEMS.

Express the operation of calculating the electro-chemical equivalent of cupric copper. *Ans.* $31.55 \times 0.000{,}104{,}38 = 0.003{,}293$.

With what factor must the electro-chemical equivalent of any element be multiplied to give grams per hour?

Ans. As 1 ampere $= \frac{1}{10}$ C.G.S. unit, the factor is $3{,}600 \div 10 = 360$.

If 0.0397 gram of silver is precipitated by a current, what is the quantity of electricity in C.G.S. units and in coulombs?

Ans. 3.55 C.G.S. units; 35.5 coulombs.

Using the reciprocal of the electro-chemical equivalent of gold, 146.87, express the operation of determining the quantity of electricity required to precipitate 0.0781 gram of gold.

Ans. $0.0781 \times 146.87 = 11.47$ C.G.S. units; 114.7 coulombs.

How long will it take 3.4 amperes to precipitate 1.751 grams of cupric copper? *Ans.* 26 minutes 4 seconds.

A current precipitates from a solution of copper sulphate 0.115 gram of copper; from a second solution in series with the first the same current precipitates 0.23 gram of copper. What is the second solution? *Ans.* A solution of cuprous copper.

The same current precipitates from one solution 0.071 gram of silver and from another solution in series with the first 0.0658 gram of another metal. What is the other metal? *Ans.* Mercuric mercury.

What weight of gold will 3.75 amperes deposit in 5 hours 21 minutes? *Ans.* 49.18 grams; 758.94 grains.

What proportion will give the relative weights of cupric copper and gold precipitated by the same current, taking gold as unity?

Ans. $652 : 316 :: 1 : x = 0.484$. For each grain of gold 0.484 grain of copper will be deposited.

What weight of lead will be deposited by 3.9 amperes in 10 seconds?

Ans. 0.042 gram; 0.648 grain.

A current deposits 1.75 grams of silver in 1 hour. Calculate its strength. *Ans.* 0.4348 ampere.

The atomic weight of a pentad element is 14.01. What is its chemical equivalent? *Ans.* 2.802.

A current passing through a solution deposits 0.00231 gram of cupric copper each second. The e.m.f. expended is 1.22 volts. Calculate the power or activity. *Ans.* 8.55 watts.

A current passing through an electrolytic voltameter and then through a calorimeter deposits 16.1 grams of silver per hour and develops 6,912 calories. What e.m.f. was expended in the calorimeter? *Ans.* 2 volts.

The current in the above case is increased so as to precipitate 18 grams of silver per hour. What will be the effect on the calorimeter? *Ans.* It will show 7,728 calories.

A current of electricity is passed through two solutions, one of silver and one of cupric copper. After a time it deposits 5 grams of silver. How much copper will be deposited? *Ans.* 1.473 grams.

CHAPTER XII.

FIELDS OF FORCE.

Fields of Force. — The Unit Field. — Intensity of Field. — Polarity. — Quantity of Field. — Lines of Force per Square Centimeter and per Square Inch. — Kapp's Unit. — Fields of Uniform and of Varying Strength. — Radiant Fields of Force. — Reciprocal-Action in Fields of Force.— Induction of E.M.F. and Current.

Fields of Force. A field of force exists in any region in which, without physical contact, force is exercised upon a mass or other physical quantity. Thus when a magnet exercises force upon a piece of iron, attracting it without any contact between the two, a magnetic field of force is shown to exist in the space between the two, because magnet and iron are drawn together by a force exerted in the space in question without any physical contact or connection between the two. The force exerted in a field of force may be one of repulsion, as when two similar magnet poles repel each other.

It is known that every mass attracts every other mass without physical contact or connection. This fact shows the existence of a field of force, which is called the field of force of gravity or of gravitation. The electrician is principally concerned with the electro-magnetic and electrostatic fields of force, especially with the electro-magnetic field.

Like other physical quantities there are two things to be specified or measured in a field of force — its total value and its intensity. The unit of measurement for both of these is usually the line of force. For intensity the lines of force per unit area are to be specified, as 100 lines to the square inch or 15.5 lines to the square centimeter.

The Unit Field. Intensity of Field. A unit field of force is one which acts with unit force on a unit quantity. Thus a

unit electro-magnetic field acts on a unit quantity of magnetism with unit force, which force is a dyne. A field is said to vary in strength or intensity as it acts with more or less force on any given quantity. A unit field of force has 1 line of force to the square centimeter.

Another unit of strength of the electro-magnetic field is the gauss, a unit field being of one gauss intensity. A gauss indicates a strength of one line of force to the square centimeter.

Example. A magnet pole of 1.339 units strength is acted on by an electro-magnetic field with a force of 39.7 dynes. What is the strength of field?

Solution. A unit pole would be acted on in the same field with a force of $39.7 \div 1.339 = 29.65$ dynes. The field is of 29.65 gausses intensity or strength or of 29.65 lines of force to the square centimeter.

Example. With what force will a 12-gauss field act on a magnet pole of 5.5 units strength?

Solution. With $12 \times 5.5 = 66$ dynes.

Polarity. A field acts to attract or repel quantities subject to its action. The direction in which these actions are exerted is the polarity of the field. If a magnet needle is pivoted in a magnetic field it will take a position coinciding with the polarity of the field at the place it occupies and lying parallel with the lines of force. If a conductor is said to be moved across a field at right angles to the lines of force, it would be moved at right angles to the longitudinal axis of the magnet. A weight is attracted to the earth; the direction of its fall indicates the polarity of the earth's gravitation.

Quantity of Field. The intensity of a field multiplied by its cross-sectional area gives a quantity of field. Thus if a field is of 5 units strength, an area of 3 square cm. will include what may be called a quantity of 15 units. The calculations referring to electric fields are generally done in lines of force, and as a unit

field is one of 1 line to the square centimeter the above quantity is simply a total of 15 lines of force. Sometimes a line of force is called a weber.

Example. A field of force of 19 lines to the square centimeter acts upon a unit magnet pole. What is the force in dynes exerted upon the pole?

Solution. As a line of force is equal to 1 dyne acting upon unit quantity, and as the magnet pole is a unit pole, the answer is 19 dynes.

Example. An electro-magnet attracts a magnet pole of 31 units strength with a force of 51 grams. What is the strength of field?

Solution. 51 grams are equal to $51 \times 981 = 50{,}031$ dynes. This is the action of the field upon a quantity of magnetism of 31 units. Its action upon a unit quantity would be $\frac{1}{31}$ as great, or would be equal to $50{,}031 \div 31 = 1{,}614$ lines of force to the square centimeter.

Example. Expressing the strength of the earth's gravity in gravitational lines of force, how many lines of force to the square centimeter would it contain?

Solution. The earth's gravitational field acts upon a unit mass, 1 gram, with a force of 981 dynes (approximately). Its strength is therefore 981 lines of force to the square centimeter.

Lines of Force per Square Centimeter and per Square Inch. The lines of force of a field are often referred to the square inch. A square inch is equal to 6.45 square cm.; therefore the number of lines of force per square centimeter is reduced to lines per square inch by multiplying by 6.45. A square centimeter is equal to 0.155 square inch; so that the number of lines of force per square inch is reduced to lines per square centimeter by multiplying by 0.155.

Example. The field of a dynamo has 5,000 lines per square centimeter. Calculate the lines per square inch.

Solution. Multiplying as above we have $5{,}000 \times 6.45 = 32{,}250$ lines per square inch.

Example. Reduce 121,300 lines per square inch to lines per square centimeter.

Solution. Multiplying by the reduction factor gives $121{,}300 \times 0.155 = 18{,}801$ lines per square centimeter.

Kapp's Unit. The Kapp line of force is equal to 6,000 C.G.S. lines of force to the square inch and consequently to 930 lines of force to the square centimeter. It is a unit which is but little used.

Example. How many Kapp lines are there in a field 10 square inches in section with 3,906 lines per square inch?

Solution. The total number of lines is $3{,}906 \times 10 = 39{,}060$. Dividing this by the equivalent, 6,000, we obtain 6.51 Kapp lines.

Example. Calculate the Kapp lines in a field of 27 square cm. section with 2,750 lines to the square centimeter.

Solution. Proceeding as above, but using the other equivalent, we have $2{,}750 \times 27 = 74{,}250$ as the total number of lines, and dividing by 930, the equivalent for square centimeters, gives 79.8 Kapp lines.

Fields of Uniform and of Varying Strength. A field of force may be of uniform strength in all its parts or may vary according to any law. The lines of force of a uniform field are parallel to each other. The earth's field of gravitational force is uniform for terrestrial distances, if we exclude from consideration the effects of the compression at the poles and the effects of its rotation.

Example. Compare the attraction of the earth for the same mass at the height of 1 and of 100 cm. from its surface.

Solution. The field of force within such small distances is uniform; the attraction is therefore the same. The weight will be the same at both points.

Radiant Fields of Force. A field of force established by a point such as a magnet pole of very small size varies in strength in accordance with the laws of radiant or central forces. If two points such as minute magnet poles act upon each other the action will vary inversely with the square of the distance separating them and with the product of the strength of the poles. The force exerted upon each other by two points such as magnet poles is expressed by the following formula

$$F = N\frac{mm'}{l^2},$$

in which F is the force, N is a constant, m and m' are the quantities acting on each other, and l is the distance separating them.

In the case of magnet poles N is unity; in the case of the attraction of gravitation between two masses of small size N has a value of 6.6576×10^{-8}. In all cases where units of the C.G.S. system are used F will be determined in dynes. Thus a mass of 1 gram is attracted by another mass of 1 gram at a distance of 1 cm. by a force of 6.6576×10^{-8} dyne.

Example. A mass of 7 grams and a mass of 3 grams are 11 cm. apart. What is the amount of gravitational force they exert upon each other?

Solution. Owing to the attraction of gravitation they will attract each other. The amount of the attraction in grams is given by the formula, in which the values of the problem are substituted for the letters of the expression:

$$F = 6.6576 \times 10^{-8} \times \frac{7 \times 3}{(11)^2} = 6.6576 \times 10^{-8} \times 0.173$$
$$= 1.15 \times 10^{-8} \text{ dyne.}$$

Reciprocal Action in Fields of Force. The numerator of the fraction $N\frac{mm'}{l^2}$ expressing the value of the force consists of the product of two quantities and a constant. If either quantity m or m' is reduced to zero, the value of the fraction

will also become zero and no force will be exercised. All such force is exerted reciprocally. The earth attracts all things, but all things attract the earth. A magnet not only attracts its armature, but its armature attracts it.

In the case of an electric field m and m' are quantities of electricity or of magnetism, such as magnet poles.

Example. Two equal magnet poles act upon each other with a force of 3.4 dynes. The distance between them is 2.5 cm. What is the strength of each?

Solution. By the conditions of the problem, $m = m'$, and the formula becomes $F = \frac{m^2}{l^2}$. Substituting for F and l their values we have

$$3.4 = \frac{m^2}{(2.5)^2}, \text{ or } m^2 = (2.5)^2 \times 3.4 = 21.25 \text{ and } m = 4.6.$$

Each pole therefore is equal to 4.6 unit magnet poles.

Example. Two equally charged disks repel each other with a force of 10.4 dynes, the distance between them being 3.1 cm. Calculate the amount of charge on each, assuming them to be so small in proportion to the distance separating them that the laws of radiant action apply.

Solution. Substituting in the formula as before gives

$$10.4 = \frac{m^2}{(3.1)^2}, \text{ or } m^2 = (3.1)^2 \times 10.4 = 99.944 \text{ and } m = 10.$$

Each disk is charged with 10 electrostatic units of electric quantity.

Example. One of the preceding disks is placed 3.4 cm. from another similarly charged disk. It attracts it with a force of 2.1 dynes. What is the quantity of electricity upon the second disk?

Solution. This is a case for the formula $F = \frac{mm'}{l^2}$. We have $m = 10$, $F = 2.1$, and $l = 3.4$. Substituting as before we have

$$2.1 = \frac{10\,m'}{(3.4)^2}, \quad 10\,m' = (3.4)^2 \times 2.1 = 24.276, \text{ and } m' = 2.43.$$

The new disk is charged with 2.43 electrostatic units of quantity.

Example. A magnet pole attracts another magnet pole at a distance of 1 cm. with a force of 3 dynes. The one pole is a unit pole. What is the strength of field at unit distance from the other pole?

Solution. Here unit quantity is attracted with a force of 3 units. The field at the unit pole, therefore, has a strength of 3 units, or 3 lines of force per square centimeter.

Induction of E.M.F. and Current. If a unit length of conductor is moved at unit rate across a unit field, a unit potential difference or e.m.f. will be maintained between its ends. The conductor is assumed to move at right angles to the polarity of the field.

Example. An electro-magnetic field of force has 21 lines to the square centimeter. A wire 121 cm. long is moved across it at the rate of 32 cm. per second. What e.m.f. will be generated?

Solution. The wire is 121 units long, and moves at 32 units rate; in a unit field it would have $121 \times 32 = 3{,}872$ units of e.m.f. impressed upon it. But the field is of 21 units intensity; therefore the e.m.f. impressed would be $3{,}872 \times 21 = 81{,}312$ C.G.S. units, which are equal to 0.000,813 volt.

Current actuated by e.m.f. thus produced is induced current. For a current to be produced there must be a closed circuit. If the quantity of field included within a circuit of wire or other conductor is changed, a current will be produced in the conductor. The current is due to e.m.f. impressed. The current and e.m.f. will both vary with the rate of change of quantity of field. A unit rate of change will impress unit e.m.f. on the circuit. The unit of rate of change is the product of a cross-sectional area of 1 cm. by an intensity of 1 line of force per square centimeter per second.

Example. A circle of wire is 1 meter in diameter. Within it there is a field of 257 lines of force to the square centimeter. In 9 seconds it is reduced to 61 lines. What e.m.f. will be impressed on the circuit?

Solution. The area of the circle is 7,854 square cm. This multiplied by the rate of change of strength of field gives the e.m.f. The rate of change is the number of lines of force per square centimeter destroyed per second, or $(257 - 61) \div 9 = 21.78$. $7{,}854 \times 21.78 = 171{,}060$ C.G.S. units $= 0.001{,}710{,}6$ volt.

1 E.M. unit of e.m.f. is produced by the cutting of 1 line of force per second, or by a change in the number of lines of force within a circuit at the rate of 1 line of force per second.

1 volt e.m.f. is produced by the same process, but with 10^8 lines substituted for 1 line in the law.

Example. A field of force has a strength of 39 lines per square centimeter. A conductor 11 cm. long moves across it at the rate of 17 cm. per second. What e.m.f. will be produced?

Solution. The lines of force cut per second by the conductor are equal to the area it sweeps through in a second multiplied by the number of lines in a square centimeter, which gives

e.m.f. $= 11 \times 17 \times 39 = 7{,}293$ C.G.S. electro-magnetic units $= 0.000{,}072{,}93$ volt.

Example. A car goes at the rate of 30 miles an hour on a railroad of 4 feet 8 inches gauge. The vertical component of the earth's magnetic field may be taken as 0.438 line of force. What e.m.f. will be developed in an axle?

Solution. A speed of 30 miles an hour is 1,341 cm. per second. The length of the car axle, 4 feet 8 inches for the purposes of the problem, is 142.24 cm. Multiplying these together gives the area through which the axle sweeps in a second. It is $1{,}341 \times 142.24 = 190{,}743$ square cm. Each square centimeter has its field of 0.438 line of force. Therefore $190{,}743 \times 0.438 = 83{,}545$

is the number of lines of force cut per second and also the number of C.G.S. units of e.m.f. As a volt is equal to 10^8 C.G.S. units, this is equal to $83,545 \div 10^8 = \frac{1}{1200}$ volt, approximately.

A dynamo's or motor's working field is the portion swept through by the wires or conductors on the armature. In the case of a two-pole machine with a drum armature the conductors pass through the field twice in each revolution, but as they are connected so as to work in parallel half in parallel with half, the e.m.f. is due to the cutting of the lines of force of the field once for each revolution.

A convenient formula for the e.m.f. of such machines is the following:

$$\text{e.m.f. in volts} = \frac{nSN}{10^8},$$

in which n is the revolutions of the armature per second in a two-pole machine, or equivalent thereto in a multipolar machine, S is the number of conductors in series on the armature, and N is the number of lines of force in the field cut by the armature conductors in a revolution.

In a multipolar machine as usually wound for direct current the above value is to be multiplied by one half the number of poles if N is taken as the field due to one pair of poles. This is on the basis that a multipolar machine is usually an aggregation of bipolar ones. For such a multipolar machine the formula becomes

$$\text{e.m.f. in volts} = \frac{pnSN}{10^8},$$

in which p is the number of pairs of poles.

Example. From the pole pieces of a two-pole dynamo 7,170,000 lines of force run to the armature; there are 120 turns of wire on the armature, all in series; the machine turns at the rate of 780 revolutions per minute. Calculate the e.m.f.

Solution. 780 revolutions per minute are 13 per second. Substituting in the formula gives

$$\text{e.m.f. in volts} = \frac{13 \times 120 \times 717 \times 10^4}{10^8} = 111.9 \text{ volts.}$$

Example. A dynamo has eight poles; there are 9,000,000 lines of force in each of the fields; the armature rotates at a rate of 20 revolutions per second; there are 90 conductors on the armature, all in series. Calculate the e.m.f.

Solution. Substituting in formula (2) we have

$$\text{e.m.f. in volts} = \frac{4 \times 9 \times 10^6 \times 20 \times 90}{10^8} = 648 \text{ volts.}$$

In such examples as these the use of exponential notation simplifies the work. Thus on inspection the first of the above examples reduces to $13 \times 12 \times 0.717$ and the second to $4 \times 9 \times 2 \times 9$.

Multipolar machines are sometimes so constructed that the calculation follows the rule for bipolar machines, the lines of force in one of the fields being taken as the value of N.

PROBLEMS.

A field of 363 square cm. area contains 76,321 lines of force. What is its strength in gausses? *Ans.* 210 gausses.

A field of force has a strength of 1,100 lines to the square centimeter; its cross-sectional area is 36 square cm. How many lines of force does it contain? *Ans.* 39,600.

A field of force is specified as of 2,700 gausses and 297,000 webers. What is its cross-sectional area? *Ans.* 110 square cm.

What is the strength of field in square-inch measure if it has 4,000 lines of force to the square centimeter?

Ans. 25,800 lines of force to the square inch.

A field of force has 2,000 lines of force and is of 5 square cm. area. What will its action on a magnet pole be?

Ans. 400 dynes; 0.408 gram; 6.30 grains.

An electro-magnet attracts a magnet pole of 29 units strength with a force of 5 grams. What is its strength of field?

Ans. 169 lines of force to the square centimeter; 1,090 lines of force to the square inch.

Two equal magnet poles attract each other with a force of 0.005 gram; they are 0.5 cm. apart. What is the strength of each?

Ans. 1.107 unit poles each.

Two disks attract each other with a force of 0.003 gram, each carrying the same static charge; they are 0.7 cm. apart. How many E.S. units of quantity does each one carry? *Ans.* 1.208 units.

A magnet pole of 6 units strength attracts another one with a force of 0.0414 gram at a distance of 2 cm. What is the strength of the other magnet? *Ans.* 27.07 units.

A magnet of 7.0 unit poles is attracted at a distance of 1.5 cm. by another magnet with a force of 0.31 gram. What is the field at a distance of 1 cm. from the other magnet?

Ans. 97.75 lines of force to the square centimeter; 630.5 lines of force to the square inch.

A two-pole dynamo has 9,000,000 lines of force in its field; there are 140 conductors in its armature. At 1,200 revolutions per minute what is its voltage? *Ans.* 252 volts.

The area of the pole face of a two-pole dynamo is 36 square inches; it is excited to 65,000 lines of force to the square inch; the armature has 200 turns of wire and revolves 1,500 times per minute. Calculate the e.m.f. *Ans.* 117 volts.

There are 120,000 lines of force to the square inch in the field of an eight-pole dynamo; each pole face is 48 square inches area; on the armature there are 2,400 conductors in 8 parallel groups; the revolutions are 10 per second. Calculate the e.m.f. *Ans.* 691.2 volts.

An e.m.f. of 200 volts is to be obtained from a two-pole machine whose pole faces are of 9 square inches area each; the induction is 100,000 to the square inch; the rotation is to be 25 per second. How many conductors should there be on the armature?

Ans. 889 conductors (by computation).

CHAPTER XIII.

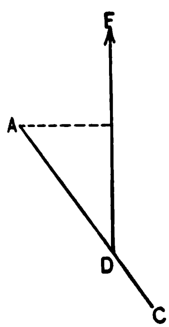
MAGNETISM.

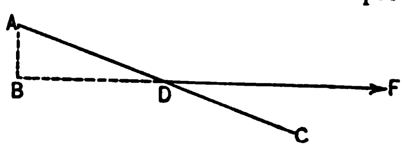
Moments. — Lever Arm of a Force. — The Couple. — Unit of Moment. — The Magnetic Filament. — Lines of Force in a Filament. — Magnetic Quantity and Strength of Pole. — Measure of Magnetic Quantity. — Lines of Force produced by Unit Quantity of Magnetism. — Moment of a Magnet. — Intensity of Magnetization. — Two Definitions of Intensity of Magnetization. — Turning Moment of a Magnet. — Magnetic Traction. — Problems.

Moments. If a bar or lever is pivoted at a point so as to turn with the point for a center, and if a force is applied to the bar at a distance from the center and at an angle to the bar so as to tend to turn it, the turning action of the force is termed its moment.

The moment of a force is the product of the force by the length of the line connecting the center of rotation with the line passing through the point of application of and in the direction of the force and perpendicular thereto. If necessary, the line passing through the point of application of the force is prolonged.

In the cut let A be the point around which as a center the lever AC is free to rotate; let the line DF represent the force applied at the point D. Drawing the line AB perpendicular to the line of the direction of the force DF, the product of the force by the length of the line AB is the moment of the force. The next cut shows the same condition of things, except that the line of the force has to be prolonged.



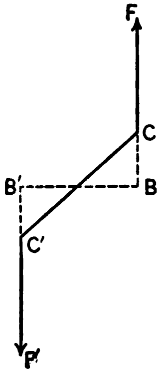


Similar letters are used to indicate the same things. DF is the force applied to the lever, AC rotating around the

point A, and AB is the lever arm of the force, and the product of AB by DF is the moment of the force.

Lever Arm of a Force. In the discussion of moments the perpendicular distance from the center of rotation to the line of the force or to its prolongation is called the lever arm of the force. In the cuts the lines AB are the lever arms of the forces DF.

The Couple. When two forces concur to produce rotation in the way described above for a single force they are termed a couple. Sometimes the term 'couple' is restricted to the two equal and opposite forces as shown in the cut, F and F', whose lever arms are AB and AB'. Such restricted use of the term applies to the action of the two magnet poles of the same magnet when the magnet is mounted so as to be free to turn about its center.



Unit of Moment. Unit of moment is a unit of force acting with a lever arm of unit length. When a force has a moment of say 10 units it may itself be of 1 dyne value with a lever arm of 10 om., or of 10 dynes value with a lever arm of 1 cm., or any relation may exist which gives the product 10.

The Magnetic Filament. A magnetic filament is a theoretical conception. If a thin filament of iron were magnetized under such conditions that no lines of force leaked out of its sides, and if it were so long that the poles would not act upon each other, it would be a magnetic filament. This condition is approximately present in the central filaments, which may be assumed to exist in and to constitute the central part of the iron or steel of a magnet.

Lines of Force in a Filament. If an isolated magnet pole be assumed to be a possible thing, it would radiate lines of force

symmetrically in all directions like the radii of a sphere. The poles of a magnetic filament are assumed to do this and to that extent to represent isolated magnet poles as shown in the cut.

Assuming the lines of force to radiate symmetrically in all directions from the magnet pole, the number of lines of force emerging from the pole are equal to the number of lines per square centimeter at the unit distance from the pole multiplied by the surface of the sphere of unit radius. From geometry we know that the surface of a sphere of radius 1 is equal to 4π. This is because the area of a sphere of radius r is equal to $4\pi r^2$. But the square of 1 is 1, and if we substitute for r the radius of the sphere in question we shall have $4\pi 1^2$, which is equal to 4π. The strength of the pole is measured by the lines of force per square centimeter at unit distance. Therefore the lines of force emerging from the pole are equal in number to the product of the strength of the pole by 4π, which is the product of $4 \times 3.1416 = 12.5664$.

An indefinitely thin filament magnetized everywhere in the direction of its length is called a magnetic solenoid. It follows that the total number of lines of force in a unit magnetic filament are 12.5664.

Example. A pole of a magnetic filament establishes a field of 10 lines of force to the square centimeter at a distance of 1 cm. from itself. What is the number of lines of force emerging from the pole?

Solution. Multiplying the lines of force per square centimeter at unit distance by 4π, we have $10 \times 12.5664 = 125.64$, the lines of force emerging from each pole after threading the filament.

Example. Assume that a field of 9 lines of force per square centimeter is established by such a pole at a distance of 3 cm. Calculate the lines of force in the filament.

Solution. Multiplying the lines of force per square centimeter at 3 cm. by the square of the distance gives the lines of force

per square centimeter at unit distance, or $9 \times (3)^2 = 81$. Then multiplying the lines of force per square centimeter at unit distance by 4π gives the answer. $81 \times 12.5664 = 1{,}017.88$.

Magnetic Quantity and Strength of Pole. Magnetic quantity is the measure of the strength of a magnet pole. The strength of a magnet pole is its attractive or repulsive force for another pole of unit strength. A pole of unit strength is one which will attract or repel another pole of unit strength at a distance of 1 cm. with a force of 1 gram. A unit pole contains 1 unit of magnetism.

A magnet pole is a center of force whence lines of force issue, passing out from it into space. A magnet pole always has a complementary pole; for a north pole there is always a south pole, and vice versa. The lines of force emerging from one pole always return to the other pole through space. Thus each line of force follows a continuous path or circuit.

Quantity of magnetism may be defined as the magnetization of a magnet irrespective of its length, but the term 'magnetization' is usually applied to the measure of the relative magnetic moment of a magnet, in which case the volume of the magnet enters as a constituent. Magnetic quantity is taken as located at the end or pole of a magnet. The unit of quantity of magnetism is the quantity which would act upon a unit pole at a distance of 1 cm. with a force of 1 dyne, so that the quantity of magnetism in a unit magnetic pole is the unit quantity of magnetism. Quantity of magnetism constitutes the strength of a magnetic pole, and the two expressions can often be used interchangeably.

Measure of Magnetic Quantity. Magnetic quantity is measured by the strength of field which it can produce at a distance of 1 cm. from its place or location, which is a magnet pole. The strength of field is measured by the dynes of force which it can exert upon a unit magnet pole. Hence the dynes of force

with which a unit pole is acted on by a quantity of magnetism 1 cm. distant give the units of magnetic quantity.

Lines of Force produced by Unit Quantity of Magnetism. Recurring to the definition of a line of force we see that the unit of magnetic quantity produces a field of one line of force per centimeter at a distance of 1 cm. from the pole where it is located.

Example. A magnet produces a field of 3 lines of force per centimeter at $\frac{1}{2}$ cm. distance from its pole. What is the quantity of magnetism in a pole?

Solution. At 1 cm. from the pole it would produce a field of $\frac{3}{4}$ line of force per centimeter. The quantity of magnetism is $\frac{3}{4}$ of a unit.

The reduction factor 4 used in the last problem is the square of the ratio of $\frac{1}{2}$ to 1; this ratio is 2, and its square is 4.

Example. A unit pole is placed at a distance of 3 cm. from a pole of unknown quantity of magnetism. The attraction between them is 9 dynes. What is the quantity of magnetism in the pole of unknown strength?

Solution. In the formula for radiant force, force $= \frac{mm'}{(\text{distance})^2}$, substitute the values of the problem. This gives

$9 = \frac{1 \times m'}{9}$, whence we find $m' = 81$, which is the quantity of magnetism in the pole. The pole of unknown strength has the strength of 81 unit poles. If it were placed at a distance of 1 cm. from a unit pole the attraction between them would be 81 dynes.

Example. Two magnets act upon each other with a force of 3.6 dynes at a distance of 1.1 cm. One of the poles acts upon a unit pole with a force of 2 dynes at a distance of 2.1 cm. What is the strength of the poles?

Solution. The general formula can be transposed from

$$\text{Force} = \frac{mm'}{(\text{distance})^2},$$

in which m and m' are the quantities acting on each other, to

$$m = \frac{\text{force} \times (\text{distance})^2}{m'}.$$

Substituting for force 2, for distance 2.1, and for m' 1, it becomes the case of the one pole acting upon the unit pole, giving

$$m = \frac{2 \times (2.1)^2}{1} = 8.82,$$

which is the strength of one of the poles. Again applying the formula with this as the value of m' we have

$$m = \frac{3.6 \times (1.1)^2}{8.82} = 0.494,$$

which is the strength of the other pole.

Example. What strength of field would be established by a 1.75 pole at a distance of 0.33 cm.?

Solution. The strength varying inversely as the square of the distance, the proportion obtains:

$$(0.33)^2 : 1 : : 1.75 : x = 16.07.$$

At 1 cm. distance from the pole there are 1.75 lines of force to the square centimeter. At 0.33 cm. distance there are 16.07 lines. At the lesser distance a unit magnet pole would be acted on by 16.07 dynes.

Moment of a Magnet. The moment of a magnet is the product of the strength of one of its poles by the distance separating the two poles, which is the same as the sum of the products of the strength of each pole by its distance from the center of the distance separating the poles.

Example. The poles of a magnet are of 12.9 dynes strength each; they are separated by 14 cm. from each other. Calculate the moment of the magnet.

Solution. It is $12.9 \times 14 = 180.6$.

Another definition of the moment of a magnet is a quantity which when multiplied by the intensity of a uniform field gives the couple which the magnet experiences when held with its axis perpendicular to the lines of force in this field.

Example. A magnet is pivoted so as to be able to turn with freedom. It is held in a field of force of 30 lines to the square centimeter, and at right angles to the lines of force. It tends to turn with a moment of 21. What is its moment?

Solution. It is 0.7, because $30 \times 0.7 = 21$.

Example. If it had a strength of pole of 0.25 dyne, how far apart would the poles be?

Solution. Its moment being 0.7, its length from pole to pole is $0.7 \div 0.25 = 2.8$ cm.

The turning effort, couple, or moment of a magnet at right angles to parallel lines of force of a unit field is the same as if each pole were acted on by a unit magnet pole at unit distance or at one centimeter and placed in a line perpendicular to the magnet. The action of a field of n lines is equal to that of a pole of quantity of magnetism n at a distance of 1 cm., or to that of a pole of quantity na^2 at a distance a.

Intensity of Magnetization. If a magnet were cut into pieces, the quantity of magnetism in the pieces would be the same as in the original magnet. The sum of the moments of the pieces would also be the same as the moment of the original magnet.

The intensity of magnetization is the magnetic moment of a unit volume of the magnet under consideration. If a magnet however large were cut into pieces of unit volume each, the sum of the magnetic moments of the pieces would be equal to the magnetic moment of the original magnet. Intensity of magnetization is the quotient of moment divided by volume. It expresses the relative strength of a magnet. It is sometimes called magnetization.

Example. A steel bar magnet weighs 453.6 grams. Its specific gravity is 7.85. Its moment is 10,088 C.G.S. units. What is its intensity of magnetization?

Solution. The weight in grams divided by its specific gravity gives the volume in cubic centimeters. Hence $453.6 \div 7.85 = 57.78$ is the volume of the bar. The quotient of the moment divided by the volume gives the intensity of magnetization, or $10{,}088 \div 57.78 = 174.6$, which is the intensity of magnetization.

Intensity of magnetization is also equal to the quantity of magnetism per square centimeter of cross section of the magnetized piece. Thus let m denote the quantity of magnetism upon a pole of cross section S, and let l denote the distance from pole to pole. The moment of the magnet is then ml. Its volume is Sl. Dividing moment by volume to obtain the moment per unit volume, which as we have seen is the intensity of magnetization, we have

$$\text{Intensity of magnetization} = \frac{ml}{Sl} = \frac{m}{S},$$

which is the magnetic quantity per unit area of cross section of the bar.

For the above to be true the magnet must be magnetized everywhere in the direction of its length. Such distribution of magnetism is termed solenoidal.

Example. A bar magnet is 1.2 cm. in diameter, 16 cm. long, and is of 90 dynes polar strength at a distance of 1 cm. Calculate the intensity of magnetization (*a*) by the magnetic quantity per unit area of cross section and (*b*) by the moment of unit volume.

Solution. The cross-sectional area of the bar is $0.6^2 \times 3.1416 = 1.13$ square cm. The intensity of magnetization is the quotient of the quantity of magnetism, 90, divided by the area of the surface over which it is distributed, 1.13, or $\frac{90}{1.13} = 79.64$.

The moment of the bar is the product of the length, 16, by

the quantity of magnetism on one pole, 90, giving $16 \times 90 = 1,440$. The volume is the product of the area, 1.13, by the length, 16, giving $1.13 \times 16 = 18.08$ cubic cm. Dividing moment by volume we have

$$\frac{1,440}{18.08} = 79.64, \text{ as before.}$$

Two Definitions of Intensity of Magnetization. Intensity of magnetization is thus definable from two standpoints, that of magnetic quantity and that of magnetic moment. Referred to magnetic quantity it is equal to the quantity in a magnet pole divided by the area of the face of the pole. It is therefore equal to the number of units of quantity or of unit poles per square centimeter of the area of the pole. This area is the end face of the magnet at right angles to the axis. Referred to moment it is equal to the moment of a cubic centimeter of the magnet. As we have seen, one of these expressions is equal to the other.

Turning Moment of a Magnet. If a magnet is turned so as to be at an angle with the lines of force of a field, it will tend to place itself parallel to the lines of force. The couple is equal to the sum of the turning moment of the two poles. As the poles are of equal strength and are generally symmetrical, the couple is numerically equal to twice the turning moment of one pole. Referring to what has been said about couples and moments (pp. 166, 167) it will be seen that the couple of a magnet in a field of force is numerically equal to the product of the strength of one pole by the length of the magnet by the sine of the angle between the axis of the magnet and the lines of force.

As the sine of 90° is 1, the couple of a magnet at right angles to the lines of force of a field is equal to the product of the strength of one pole by the length of the magnet. In other words, the couple of a magnet at right angles to a field of force, which

means at right angles to the lines of force of such field, is equal to the moment of the magnet.

Example. A magnet held at right angles to a unit field of force has a couple of 22. What is the strength of one pole in dynes if it is 25 cm. long?

Solution. The magnet's length being 25 cm., the moment is 25 times as great as if the magnet were 1 cm. long. The strength of one pole is therefore $\frac{22}{25} = 0.88$. To prove the correctness of the method refer to the law that the moment of a magnet is equal to the product of the strength of one pole by the length of the magnet. The above operation reverses the steps indicated by this law.

Example. If the above magnet were placed as described in a field of force of a strength of 3.7, what would the couple be?

Solution. It would be the product of the pole strength by the length of the magnet by the strength of the field. $0.88 \times 25 \times 3.7 = 8.14$.

Example. A magnet is placed in a field of force at an angle to the lines of the field of 45°. The magnet is 2.8 cm. long. What is the lever arm of its couple?

Solution. Accurately speaking there are two arms. Each is equal to the length of one half of the magnet multiplied by the sine of the angle which it makes with the lines of the field. The sine of 45° is 0.7071 and the length of one arm of the magnet is $2.8 \div 2 = 1.4$. $1.4 \times 0.7071 = 0.99$, which is the lever arm of each pole. The couple is equal to the sum of the turning moments of the two poles. The turning moment of a pole is the product of its strength into the lever arm of its action, in this case 0.99.

Magnetic Traction. The formula for the traction between a magnet and its armature when the two are in contact is deduced on pages 192–194. It is

$$\text{Traction} = \frac{AB^2}{8\pi},$$

in which A is the area of the face in contact with the armature and B is the lines of force that pass through one square centimeter of the area of contact.

Example. A magnet with a face area of 1.1 square cm. and 250 lines of force to the square centimeter is in contact with an armature. Calculate the traction.

Solution. Applying the formula we have

$$\text{Traction} = \frac{1.1 \times (250)^2}{8 \times 3.1416} = 2{,}736 \text{ dynes} = 2.789 \text{ grams.}$$

PROBLEMS.

What is the number of lines of force in a magnetic filament which produces a field of force of 11 lines of force to the square cm. at a distance of 3.3 cm. from one of its poles? *Ans.* 1,505 lines of force.

A filament has in it 90 lines of force. What field will it produce at 1 cm. from one of its poles?

Ans. 7.162 lines of force to the square centimeter.

What quantity of magnetism is in the magnet of the last problem?

Ans. 7.162 units.

If a magnet had 19 units of magnetism, what field would it produce at a distance of 19 cm. from one of its poles?

Ans. 0.526 lines of force to the square centimeter.

What is the magnetic quantity of a magnet pole if the magnet attracts a 17-unit pole at a distance of 5 cm. with a force of 3 dynes?

Ans. 4.41 units.

Two poles of unknown strength act on each other with 1.3 milligrams force at a distance of 4 cm. One (a) acts on a unit pole at a distance of 3 cm. with a force of 2 milligrams. Calculate the strength of the two poles. *Ans.* (a) 17.66 units; (b) 1.15 units.

Calculate the moment of a 10-inch magnet with 10 milligrams strength of pole. *Ans.* 249.2 C.G.S. units.

A magnet in a field of force of 91 lines of force to the square centimeter at right angles to the lines tends to turn with a force of 82 dynes at each pole; it is 25.4 cm. long. What is its moment?

Ans. 22.9 C.G.S. units.

A magnet is 0.3 square cm. in cross-sectional area, is 20 cm. long, and at 1 cm. distance attracts a unit pole with a force of 3 grams. Calculate its intensity of magnetization by two methods.

Ans. (*a*) 2,943 dynes ÷ 0.3 = 9,812 C.G.S. units. (*b*) Moment = 58,860; volume = 6 cubic cm.; 58,860 ÷ 6 = 9,810 C.G.S. units.

A magnet is 12 cm. long; it is placed in a field of force of 11 lines to the square centimeter at an angle of 30° to the lines of force; it has 30 units of magnetism. What is the couple?

Ans. 1,980 C.G.S. units.

If a field of 300 lines of force to the square centimeter were to be replaced by a magnet of 29 units strength, at what distance from the pole would the original field exist? *Ans.* 0.31 cm.

How far from a point would a 10-unit magnet pole have to be to represent in its action at the point a 1.3 field of force? *Ans.* 2.8 cm.

What is the intensity of magnetization of a magnet weighing 395 grams, with a moment of 11,721, taking its specific gravity as 7.85?

Ans. 232.9 units.

There are 1,000 lines of force to the square centimeter in a magnet; its pole area is 1.3 square cm. When in contact what will the traction be between it and its armature? *Ans.* 51,725 dynes; 52.7 grams.

The area of contact between a permanent magnet face and armature surface is 2.9 square inches; 15,000 lines of force to the square inch pass through the area of contact. Calculate the traction in dynes, grams, and pounds.

Ans. 4,025,130 dynes; 4,103 grams; 9.05 pounds.

CHAPTER XIV.

ELECTRO-MAGNETIC INDUCTION.

Induction of Magnetism. — Relation of Induced Magnetization to Field. — Susceptibility. — Table of Susceptibility. — Magnetic Induction. — Permeability. — Permeance. — Reluctance. — Permeance and Reluctance.— Formulas for Inch Measurements. — The Magnetic Circuit. — Ampere Turns. — Intensity or Strength of Field Referred to C.G.S. Unit Turns. — Strength of Field Referred to Ampere Turns. — Total Field Referred to Ampere Turns. — Reluctance of Air. — Magneto-motive Force. — Intensity of Field at Center and Ends of Coil Interior. — Magnetic Circuit Calculations. — Reluctance of Circuit of Iron. — Ampere Turns for a Given Field. — Magnetic Traction. — Determination of Permeability from Traction. — Problems.

Induction of Magnetism. Assume a field of force to be produced in the air or in a vacuum, and let a piece of iron be placed in the field. The original field will exist in it exactly as if it were air or a vacuum, and in addition thereto magnetism will be induced in the iron, so that more magnetism will be present in a unit cross-sectional area of the iron than in a corresponding area of the field. The additional quantity of magnetism per unit cross-sectional area is termed intensity of induced magnetization, or simply induced magnetization and is indicated by $\boldsymbol{I}$. The magnetic intensity of the inducing field per unit of cross-sectional area being indicated by $\boldsymbol{H}$, the total magnetism of the iron is equal to its cross-sectional area multiplied by the total quantity of magnetism per unit area, and as the latter is the sum of $\boldsymbol{H} + \boldsymbol{I}$, if we call the area A the total magnetism in the iron will be $A \times (\boldsymbol{H} + \boldsymbol{I})$.

Intensity of induced magnetization is measured by units of magnetic quantity, one of which units is the quantity in a unit magnet pole.

Relation of Induced Magnetization to Field. — Susceptibility. The value of induced magnetization depends on the

value of the magnetizing force which induces it, exactly as the current due to e.m.f. depends on the value of the e.m.f., with one difference. The relation of magnetizing force to induced magnetization is expressed by the quotient of induced magnetization divided by magnetizing force. The quotient is called susceptibility and is indicated by κ (the Greek letter kappa) or by the letter K. The relation is expressed as far as susceptibility is concerned by the two expressions

$$\kappa = \frac{\boldsymbol{I}}{\boldsymbol{H}} \text{ and } \boldsymbol{I} = \kappa \boldsymbol{H}.$$

If κ were conductivity, $\boldsymbol{H}$ electro-motive force, and $\boldsymbol{I}$ current, the above formulas would correspond in form to Ohm's law. The analogy fails and the difference just spoken of appears because of the law that the value of κ changes for different values of $\boldsymbol{I}$.

The value of κ is also different for different irons, and has to be determined experimentally for each.

Example. A bar of iron is exposed to a magnetizing force of 1.7, and each element of 1 square cm. cross-sectional area has induced magnetization of 49.9 units induced in it. What is the susceptibility of the iron at the given excitation (49.9)?

Solution. The magnetizing force per square centimeter of field is 1.7, which is $\boldsymbol{H}$ of the formula; the induced magnetization, 49.9, is $\boldsymbol{I}$. Substituting in the formula we have

$$\kappa = \frac{\boldsymbol{I}}{\boldsymbol{H}} = \frac{49.9}{1.7} = 29.4,$$

which is the susceptibility of the iron at the given excitation.

Example. In a field of $\boldsymbol{H} = 3.5$ a pole strength of 172.2 dynes is induced in a bar of 0.3 square cm. area. What is the susceptibility of the iron of which the bar is composed?

Solution. The pole strength divided by the area of the bar is the intensity of magnetization. $172.2 \div 0.3 = 574 = \boldsymbol{I}$.

I divided by H gives the susceptibility, $574 \div 3.5 = 164$, which is the value of κ, or the susceptibility.

Table of Susceptibility. The following table gives the values of susceptibility for different values of induced magnetization in wrought iron, according to experiments by Ewing and Bidwell.

I	κ	I	κ
3	10	1,173	115
32	23	1,249	56
117	53	1,337	17
574	164	1,452	7
917	187	1,530	2.6
1,078	161	1,660	0.067

Example. What field is required to induce a magnetization of 917?

Solution. The formula $H = \frac{I}{\kappa}$ with the values of the problem and of the table substituted becomes

$$H = \frac{917}{187} = 4.9,$$

which is the magnetizing force.

Susceptibility is also called the coefficient of induced magnetization.

Magnetic Induction. The magnetic intensity of the field, as we have seen, is generally expressed in gausses or in lines of force per unit of cross-sectional area of the field, one line of force per square centimeter being the expression for the unit strength of field. The strength of field due to a magnetic filament at 1 cm. distance from its pole is equal to as many lines of force per square centimeter as there are units of magnetism in the pole. The lines of force in the filament are equal to the intensity of the field at 1 cm. distance from the pole multiplied by 4π (12.566).

If induced magnetization is multiplied by 12.566, the product will be the lines of force per square centimeter in the iron due to such induced magnetization. But there are also present in the iron the lines of force of the original field, designated by $\boldsymbol{H}$; therefore the total number of lines of force per square centimeter in the iron is the sum of these two quantities. The sum is

$$\boldsymbol{H} + 12.566\,\boldsymbol{I}.$$

This quantity is called magnetic induction and is designated by $\boldsymbol{B}$.

Example. A field of intensity 12.5 acts upon an iron core of susceptibility 98. Calculate the lines of force per square centimeter in the iron.

Solution. The induced magnetization is equal to the product of the field by the susceptibility, or $12.5 \times 98 = 1,225$. This is the value of $\boldsymbol{I}$. The lines of force per square centimeter in the iron are equal to the sum of the intensity of field and the product of the induced magnetization by 12.566, which gives

$$12.5 + (12.566 \times 1,225) = 15,406,$$

which is the value of $\boldsymbol{B}$ and is expressed in lines of force per square centimeter of cross-sectional area of the core.

Permeability. Returning to the formula for magnetic induction and remembering that $\boldsymbol{I} = \kappa\boldsymbol{H}$, the formula can be written in two ways, thus

$$\boldsymbol{H} + 12.566\,\boldsymbol{I},$$

and substituting for $\boldsymbol{I}$ its value, $\kappa\boldsymbol{H}$, and introducing the symbol $\boldsymbol{B}$,

$$\boldsymbol{B} = \boldsymbol{H} + 12.566\,\kappa\boldsymbol{H}, \text{ or } \boldsymbol{H} + 4\,\pi\kappa\boldsymbol{H} = \boldsymbol{H} \times (1 + 4\,\pi\kappa).$$

The compound factor $(1 + 4\,\pi\kappa)$ is called permeability; its reciprocal, $\frac{1}{1 + 4\,\pi\kappa}$, is called reluctivity. The symbol of permeability is μ. In engineering calculations permeability is the foundation of most of the work affecting magnetic circuits; sus-

ceptibility is the basis of the theory of permeability, but is less used in practical calculations.

Example. A field of intensity $H = 5.1$ acts upon a sample of iron of susceptibility $\kappa = 169$. Calculate the permeability and induction.

Solution. The permeability is $1 + 4\pi\kappa = 1 + (12.566 \times 169) = 2{,}125$. The induction is the product of the permeability by the intensity of field, or $2{,}125 \times 5.1 = 10{,}838$, which are the lines of force per square centimeter of cross-sectional area of the iron.

Example. What is the reluctivity of the above iron?

Solution. It is the reciprocal of the permeability;

$$\frac{1}{\mu} = \frac{1}{2{,}125} = 0.00047.$$

Permeability varies in different irons, and also varies as the lines of force per given cross-sectional area differ in number. The values of permeability for two samples of iron at different values of induction are given in the tables.

Permeance. Permeance stands in the same relation to permeability as that occupied by resistance with reference to resistivity or specific resistance. It is the power of a specified circuit or portion of a circuit for carrying lines of force. The substance carrying the lines of force is a geometrical solid. If prismatic or linear in shape the permeance will vary directly with the cross-sectional area and inversely with the length, being equal to

$$\text{Permeability} \times \frac{\text{cross-sec. area}}{\text{length}}, \text{ or } \mu\,\frac{A}{l},$$

A and l indicating cross-sectional area and length respectively.

Reluctance. Reluctance is the reciprocal of permeance, and is therefore expressed by

$$\frac{l}{\mu A}.$$

Tables of Permeability. The following tables give values of permeability for different values of magnetic field.

SQUARE CENTIMETER MEASUREMENT.

Annealed Wrought Iron.			Gray Cast Iron.		
B	μ	H	B	μ	H
5,000	3,000	1.66	4,000	800	5
9,000	2,250	4	5,000	500	10
10,000	2,000	5	6,000	279	21.5
11,000	1,692	6.5	7,000	133	42
12,000	1,412	8.5	8,000	100	80
13,000	1,083	12	9,000	71	127
14,000	823	17	10,000	53	188
15,000	526	28.5	11,000	37	292
16,000	320	50	....	....	....
17,000	161	105	....	....	....
18,000	90	200	....	....	....
19,000	54	350	....	....	....
20,000	30	666	....	....	....

SQUARE INCH MEASUREMENT.

Annealed Wrought Iron.			Gray Cast Iron.		
B	μ	H	B	μ	H
30,000	4,650	6.5	25,000	763	32.7
40,000	3,877	10.3	30,000	756	39.7
50,000	3,031	16.5	40,000	258	155
60,000	2,159	27.8	50,000	114	439
70,000	1,921	36.4	60,000	74	807
80,000	1,409	56.8	70,000	40	1,480
90,000	907	99.2	....	....	....
100,000	408	245	....	....	....
110,000	166	664	....	....	....
120,000	76	1,581	....	....	....
130,000	35	3,714	....	....	....
140,000	27	5,185	....	....	....

In a magnetic circuit there are often included portions of different permeance. The reciprocal of the sum of the reciprocals of the permeances of the different portions of the circuit is the total permeance. By using the property of reluctance in the calculations the use of reciprocals is avoided. The reluctance of a magnetic circuit is equal to the sum of and is obtained by adding together the reluctances of its parts. This operation takes the place of the one just described.

Example. A bar of iron has a permeability of 2,079. It is circular in section, 1 inch in diameter, and 16 inches long. What is its permeance?

Solution. The area of the bar is 0.7854 square inch; 1 square inch is equal to 6.45 square cm. $6.45 \times 0.7854 = 5.066$ square cm., the area in square centimeters. The length of the bar in centimeters is equal to $16 \times 2.54 = 40.64$ cm. Substituting in the formula we have

$\mu\frac{A}{l} = 2{,}079 \times \frac{5.066}{40.64} = 259.15$, which is the permeance. The reluctance is $\frac{1}{259.15} = 0.00386$.

Permeance and Reluctance. Formulas for Inch Measurements. The formula for permeance is, for centimeter measurement, $\mu\frac{A}{l}$. To reduce this to inch measurements is to put it into such form that if the length of the core is given in inches, and if the area of the core is given in square inches, it will give the permeance directly. Taking the area of a square inch as 6.45 square cm., and the length of an inch as 2.54 cm., the formula becomes, for inch measurements,

$$\frac{6.45\,\mu A}{2.54\,l} = 2.54\,\frac{\mu A}{l}.$$

Example. Calculate the permeance of a bar of iron 10 inches long and 1 square inch in area, whose permeability is 1,200 under the assumed conditions.

Solution. The quotient of the product of the cross-sectional area by the permeability divided by the length is $(1,200 \times 1) \div 10 = 120$. This multiplied by the factor 2.54 gives 304.8, which is the permeance.

Or substituting directly in the formula we have

$$\text{Permeance} = 2.54 \times \frac{1,200 \times 1}{10} = 304.8.$$

The formula for reluctance is, for centimeter measurement, $\frac{l}{\mu A}$. To reduce this to inch measurement, l and A must be multiplied by the factors which will reduce linear inches and square inches to centimeters also linear and square, as before. Introducing these factors into the reluctance formula it becomes $\frac{2.54\,l}{6.45\,\mu A} = 0.394\frac{l}{\mu A}$. This formula can be used when the dimensions are given in inches.

Example. The mean length of the core of a magnetic circuit is 18 inches; the core is circular in section and 4.3 square inches in cross-sectional area. The permeability, at the value of $\boldsymbol{H}$ employed, is 2,200. Calculate the reluctance.

Solution. The quotient of the length divided by the product of the cross-sectional area and permeability is $18 \div (4.3 \times 2,200) = 0.0019$. This has to be multiplied by 0.394, because the dimensions are given in inches. $0.0019 \times 0.394 = 0.000748$, which is the reluctance of the magnetic circuit.

Or substituting in the formula the values of l and A, it becomes

$$\text{Reluctance} = 0.394\frac{18}{2,200 \times 4.3} = 0.000,748.$$

The Magnetic Circuit. The magnetic line of force always forms a closed curve, regular or irregular in shape, and a collection of lines of force due to the same field are in a general way parallel to each other or concentric. They define a path which is called the magnetic circuit. It is in its laws analogous to the

electric circuit. The lines of force may lie within a core of iron or may be partly within the metal and partly outside of it.

The field is always produced by a coil of wire through which a current of electricity is passed. A mass of iron within the coil by its permeance increases the flow of lines of force and is called the core. This outlines the arrangement practically in universal use in the construction of generators, motors, and other machinery of that type.

Ampere Turns. The product of the number of turns in a field coil by the amperes of current passing through its wire is called the ampere turns. The strength of field varies with the ampere turns. If the current is expressed in other units, other compound units result, such as C.G.S. unit turns. Such units are little used.

Example. A field coil is wound of wire, which with its insulation is 0.160 inches in thickness. In the operation of the machine it carries a current of 9.4 amperes. The internal diameter of the coil is 0.80 inches; the external diameter is 2.80 inches; its length is 3 inches. What is the number of the ampere turns?

Solution. Half the difference of the diameters gives the thickness of the coil; it is $(2.80 - 0.80) \div 2 = 1$ inch. In an inch there are 6 turns of wire, as there can be no fractional turns. The length of the coil gives room for 18 turns, and the thickness for 6 turns, a total of turns for the coil of $6 \times 18 = 108$ turns. As the wire carries a current of 9.4 amperes, the product of $108 \times 9.4 = 1{,}015.2$ ampere turns is the answer.

Intensity of Field Referred to C.G.S. Unit Turns. The intensity of field produced by a coil whose axial length is great compared to its diameter varies at different points. It is the field in the interior of the coil that produces the magnetic circuit. It is proved that the intensity of the field in the interior of the coil at a part equally distant from each end is equal to the product

of the current in C.G.S. units by the turns in the coil and by 4π divided by the axial length of the coil. If we call the current strength in C.G.S. units i', and call the number of turns in the coil S, the above statement becomes in the shape of a formula,

$$\text{Intensity of field} = \frac{4\,\pi S i'}{l}.$$

All the above refers to a coil without a core of any magnetic substance.

Example. What is the intensity of field in the center of a coil 115 cm. long, with 550 turns, when it is passing a current of 11 C.G.S. units?

Solution. In this case $S = 550$, $i' = 11$, and $l = 115$. We will assume the coil to be of relatively small diameter. Our formula becomes

$$\text{Intensity of field} = \frac{4 \times 3.1416 \times 550 \times 11}{115} = 661.1.$$

These are the lines of force per square centimeter of cross-sectional area at the center of the coil, i.e., the dynes of force with which the field would act upon a unit magnet pole placed at a point on the axis of the coil equally distant from each end.

Strength of Field Referred to Ampere Turns. If we denote the intensity of a current in amperes by i, and take 12.566 as the value of $4\,\pi$, and remember that an ampere is equal to 10^{-1} C.G.S. unit, the formula becomes for amperes

$$\text{Intensity of field} = \frac{12.566 \times Si}{l} \div 10 = \frac{1.2566 \times Si}{l}.$$

Example. Assume the preceding case but expressing the current in amperes, and calculate the lines of force per square centimeter at the center of the coil.

Solution. We have as before, $S = 550$, $l = 115$, and the current strength, i, is 110 amperes. This is because, as 1 C.G.S.

unit is equal to 10 amperes, 11 C.G.S. units are equal to 110 amperes. Substituting in the formula gives

$$\text{Intensity of field} = \frac{1.2566 \times 550 \times 110}{115} = 661.1$$

as before.

Total Field Referred to Ampere Turns. The rules just given are for determining the relative strength of field. The total number of lines of force within the coil will obviously depend upon the cross-sectional area. If the intensity of field be multiplied by the cross-sectional area, the result will be the total number of lines of force. Instead of multiplying the expression by the symbol of the cross-sectional area, which may be A, the denominator of the fraction may be divided by it. The result is identical. For the total lines of force at the central part of the interior of a magnetizing coil this gives

$$\text{Total lines of force} = \frac{1.2566 \times Si}{l} \times A, \text{ or } \frac{1.2566 \times Si}{l/A}.$$

The expressions are given for amperes only.

Reluctance of Air. The denominator of the fractional expression last given is the reluctance of the column of air in the interior of the coil. This denominator is l/A.

Magneto-motive Force. The numerator of such expressions, $0.4\,\pi Si$, is sometimes called magneto-motive force.

Intensity of Field at Center and Ends of Coil Interior. The intensity of field at the ends of the coil is one-half as great as that at the center.

Example. A magnetizing coil is 229 cm. long; it has 4,400 turns, and a current of 22.5 amperes passes through it. The cross-sectional area of the core space of the coil is 11 square cm. Calculate the lines of force at the axial center.

Solution. Applying the formula for intensity of field gives

$$\text{Intensity of field} = \frac{1.2566 \times 4{,}400 \times 22.5}{229} = 543.2.$$

Multiplying by the cross-sectional area gives

$543.2 \times 11 = 5{,}975$ = total lines of force in the center of the field.

The numerator of the above expression may be treated as magneto-motive force and the quotient of l/A as reluctance. Thus

Magneto-motive force $= 1.2566 \times 4{,}400 \times 22.5 = 124{,}403$.

Reluctance $= 229 \div 11 = 20.82$.

Dividing magneto-motive force by reluctance gives

$$\text{Total lines of force of field} = \frac{124{,}403}{20.82} = 5{,}975.$$

Magnetic Circuit Calculations. If an iron core forming a complete magnetic circuit be placed within a magnetizing coil, and if its permeance be so high that the lines of force follow it without leakage, the magnetic flow will be uniform throughout the core. The formula given for the lines of the field within the coil at its center now applies to all parts of the core, with two differences — the mean length of the core is taken as the length l of the formula, and the area has to be multiplied by the permeability:

$$\text{Lines of force or magnetic flow} = \frac{1.2566\,Si}{l/\mu A}.$$

Example. An iron ring is 24 cm. in internal diameter; it is made of a cylindrical bar of metal 3 cm. in diameter. It is wound with a coil of insulated wire of 127 turns. A current of 3 amperes is passed through the wire. The permeability of the iron at the given value of $\boldsymbol{H}$ is 2,300. Calculate the lines of force in the ring under the supposition that there is no leakage.

Solution. The magneto-motive force is $1.2566 \times 127 \times 3 = 478.764$. The reluctance is the quotient of the mean length of the ring by the product of its cross-sectional area by 2,300. The mean length is the product of the mean diameter, $24 + 3$

= 27, by π. This product is $27 \times 3.14 = 84.78$. The cross-sectional area is the product of the square of one-half the diameter of the bar by π. 1.5 (half the diameter of the bar) squared = 2.25 and $2.25 \times 3.14 = 7.07$, the cross-sectional area of the bar. $7.07 \times 2,300 = 16,261$. This is μA. Dividing the length by this gives $84.78 \div 16,261 = 0.0052$, which is the reluctance of the ring. Finally, $478.764 \div 0.00522 = 91,711$, the lines of force which thread the iron passing around the ring.

Reluctance of Circuit of Iron. If for the denominator of the expression $\frac{1.2566\,Si}{l/\mu A}$ just given, which denominator is $l/\mu A$, and which is the reluctance of the circuit, the expression $0.394\ l/\mu A$ be substituted, the formula will apply to inch measurements. It becomes

$$\text{Lines of force or magnetic flow} = \frac{1.2566\ Si}{0.394\ \ l/\mu A} = \frac{3.19\ Si}{l/\mu A}.$$

Example. The length of a core of an electro-magnet and of its armature forming a complete circuit is 33.6 inches. The cross-sectional area is 1.1 sq. in. At a permeancy of 2,300 what is the total number of lines of force when excited by 381 ampere turns?

Solution. The ampere turns, 381, multiplied by 3.19 equal 1,215.39. This is the dividend or numerator. The length of core is 33.6, which has to be divided by the product of the area, 1.1, by the permeability, 2,300, which product is 2,530. $33.6 \div 2,530 = 0.01328$. This is the divisor or denominator of the expression. Then $1,215.39 \div 0.01328 = 91,520$, which is the magnetic flow.

We may substitute the values given directly in the formula, obtaining as the value of the magnetic flow

$$\frac{3.19 \times 381}{33.6 \div (2,300 \times 1.1)} = 91,520, \text{ as before.}$$

Ampere Turns for a Given Field. The problem most called for in electric work where the magnetic circuit is concerned is the determination of the ampere turns required to produce a given field. If we call the lines of force of the field, or the magnetic flow, F, we have from the preceding section

$$F = \frac{1.2566\ Si}{l/\mu A} \text{ for centimeter measurements.}$$

In this equation Si are the ampere turns. By dividing and transposing, we obtain

$$Si = \frac{lF}{1.2566\ \mu A}.$$

The ampere turns are equal to the quotient of the length of the core multiplied by the lines of force divided by the product of the permeability by the cross-sectional area by 1.2566.

Example. What is the number of ampere turns which would be required to produce 440,000 lines of force in a bar of iron 25.4 cm. long and 25.8 sq. cm. cross-sectional area, at a permeability of 166?

Solution. The numerator of the fraction is the product of the length of the core by the number of lines of force, or $25.4 \times 440{,}000 = 11{,}176{,}000$. The denominator is the product of the cross-sectional area by the permeability and by 1.2566, or $166 \times 25.8 \times 1.2566 = 5{,}381.76$. Dividing we have

$$11{,}176{,}000 \div 5{,}381.76 = 2{,}076,$$

which are the ampere turns required.

If in the formula $Si = \dfrac{lF}{1.2566\ \mu A}$, which is for centimeter measurements, l is multiplied by 2.54, which is the number of centimeters in an inch, and if A is multiplied by 6.45, which is the number of square centimeters in a square inch, it gives

$$Si = \frac{2.54\ lF}{1.2566\ \mu \times 6.45\ A} = 0.3133\ \frac{lF}{\mu A}.$$

Example. Calculate the ampere turns required to force 440,000 lines of force through a core of iron 10 inches long, 4 sq. in. in area, and of 166 permeability.

Solution. The product of the length by the lines of force lF is $10 \times 440,000 = 4,400,000$. This is the numerator of the fraction. The product of the permeability by the cross-sectional area μA is $166 \times 4 = 664$. Dividing we have

$$4,400,000 \div 664 = 6,626.5,$$

and multiplying by the factor 0.3133 we have

$$6,626.5 \times 0.3133 = 2,076$$

which are the number of ampere turns required.

Magnetic Traction. The traction or attractive force exerted between a magnet face and another magnet face or armature with which it is in contact is expressed by the formula

$$\text{Traction} = \frac{A\,(\boldsymbol{B}^2 - \boldsymbol{H}^2)}{8\,\pi},$$

which is thus deduced:

A magnet face may be regarded as a collection of unit magnet poles each of which is attracted by a force of 1 dyne in a unit field and by a force of $\boldsymbol{B}$ dynes in a field of intensity $\boldsymbol{B}$. $\boldsymbol{I}$ has been defined as the intensity of magnetization; its numerical value may be taken as expressing the number of unit poles in a square centimeter of cross-sectional area of the magnet core at any given place. Thus the action of a unit field on a magnet face of 1 sq. cm. area is equal to $\boldsymbol{I}$, and on a face of A square centimeter area is equal to $A\boldsymbol{I}$.

When a magnet and armature are in contact the field is made up of two component parts equal in value, each one due to one of the faces in contact. The action is therefore the product of the field due to one of the faces multiplied by the equivalent of the unit magnet poles represented by the other face.

Let the strength of field be $\boldsymbol{H}$, as usual; then as we have seen the induction will be $\boldsymbol{H} + 4\pi\boldsymbol{I} = \boldsymbol{B}$, by the usual nomenclature. The portion of the induction due to the faces of the magnet and armature is $4\pi\boldsymbol{I} = \boldsymbol{B} - \boldsymbol{H}$, and that due to a single face is $\frac{1}{2}(\boldsymbol{B} - \boldsymbol{H})$. It is obvious that this portion cannot act upon itself, so the attraction is the product of the magnetism in one face, $A\boldsymbol{I}$, acted on by the rest of the field intensity, which is $\boldsymbol{B} - \frac{1}{2}(\boldsymbol{B} - \boldsymbol{H}) = \frac{1}{2}(\boldsymbol{B} + \boldsymbol{H})$. From the equation $\boldsymbol{H} + 4\pi\boldsymbol{I} = \boldsymbol{B}$, we obtain a value of $\boldsymbol{I}$ in terms of $\boldsymbol{B}$ and $\boldsymbol{H}$, or $\boldsymbol{I} = \dfrac{\boldsymbol{B} - \boldsymbol{H}}{4\pi}$. Then $A\boldsymbol{I} \times \frac{1}{2}(\boldsymbol{B} + \boldsymbol{H}) = A \times \dfrac{\boldsymbol{B} - \boldsymbol{H}}{4\pi} \times \frac{1}{2}(\boldsymbol{B} + \boldsymbol{H}) = \dfrac{A(\boldsymbol{B}^2 - \boldsymbol{H}^2)}{8\pi}$, so that

$$\text{Traction} = \frac{A(\boldsymbol{B} - \boldsymbol{H}^2)}{8\pi}. \tag{1}$$

In the above formula the attraction is given in dynes; as a dyne is approximately $\frac{1}{981}$ gram, the formula for grams is

$$\text{Traction} = \frac{A(\boldsymbol{B}^2 - \boldsymbol{H}^2)}{24{,}655} \text{ grams,} \tag{2}$$

in which the denominator is the product of $981 \times 8\pi$.

The area of contact being expressed in inches the area A of the formula has to be multiplied by 6.45, the number of square centimeters in a square inch, to give its effect. The factor $(\boldsymbol{B}^2 - \boldsymbol{H}^2)$of the formula expressing in it the square of lines of force to the square inch has to be divided by $(6.45)^2$ to give its effect for square inch measurement. The value of traction has to be divided by 453.6, the number of grams in a pound, to reduce it to pounds. This gives a fraction $\dfrac{6.45}{(6.45)^2 \times 453.6} = \dfrac{100}{292{,}572}$. This gives for pounds traction and inch measurements,

$$\text{Traction} = \frac{A(\boldsymbol{B}^2 - \boldsymbol{H}^2)}{24{,}655} \times \frac{100}{292{,}572} = \frac{A(\boldsymbol{B}^2 - \boldsymbol{H}^2)}{72{,}133{,}627}.$$

Example. Calculate the traction per square centimeters in grams for a field of 350 lines per square centimeter, with an induction of 19,820 lines per square centimeter in the case of an electro-magnet in contact with its armature.

Solution. Substituting the values in formula (2) it becomes by introducing these values

$$\text{Traction} = \frac{19{,}820^2 - 350^2}{24{,}655} = \frac{392{,}832{,}400 - 122{,}500}{24{,}655}$$
$$= 16{,}086 \text{ grams.}$$

For a permanent magnet H is equal to 0, and the formulas become

$$\text{Traction} = \frac{AB^2}{8\pi} \text{ dynes.} \tag{4}$$

$$\text{Traction} = \frac{AB^2}{24{,}655} \text{ grams.} \tag{5}$$

$$\text{Traction} = \frac{AB^2}{72{,}133{,}627} \text{ pounds.} \tag{6}$$

Determination of Permeability from Traction. We have seen that $B = \mu H$. Substituting this value of B in formula (2) gives

$$\text{Traction} = \frac{A\,(\mu^2 H^2 - H^2)}{24{,}655} = \frac{AH^2(\mu^2 - 1)}{24{,}655},$$

whence

$$\mu = \sqrt{\frac{24{,}655 \times \text{traction (in grams)}}{AH^2} + 1}. \tag{7}$$

Example. An iron bar is placed in a field of 5.85 lines to the square centimeter. The area of its end is 5 sq. cm. and the traction it exerts in contact with another bar in the same field is 7,995 grams. Calculate the permeability of the iron.

Solution. Substituting these values in formula (7) gives

$$\mu = \sqrt{\frac{24{,}655 \times 7{,}995}{5 \times (5.85)^2} + 1},$$

whence the value of μ, or the permeability of the iron, is 1,073.

PROBLEMS.

A piece of iron is placed in a field of 2.4 lines to the square centimeter and there are produced in it 129 unit poles. What is its susceptibility?

Ans. 53.75.

If 175 units of induced magnetization are produced in a bar of iron by a magnetic field of 2.7 intensity, what is the susceptibility of the iron?

Ans. 64.8.

Express the above with conventional symbols.

Ans. $H = 2.7$; $I = 175$; $\frac{I}{H} = \frac{175}{2.7} = 64.8.$

If a sample of iron for a value of intensity of magnetization I of 1,000 has a susceptibility κ of 158, what is the intensity of field H required to impart such magnetization?

Ans. 6.33 lines of force to the square centimeter.

A piece of iron of susceptibility 100 is placed in a field of 11.2 lines of force to the square centimeter. Calculate its intensity of magnetization. *Ans.* 1,120 lines of force to the square centimeter.

In iron such as that of the table on page 180, what value of H is required for a value of $I = 574$? *Ans.* $\frac{574}{164} = 3.5.$

If the magnetic intensity of a field is 4.7 lines to the square centimeter, and the intensity of magnetization I due to such value of H is 875, what is the value of the magnetic induction B? *Ans.* 11,000.

What is the susceptibility in the above case? *Ans.* 186.2.

Let $H = 585$ and let the intensity of the induced magnetization I due to it in a piece of iron be 1,530. Calculate the value of B.

Ans. 19,811.

Let $H = 23{,}500$ and $I = 1{,}600$. Calculate the value of B.

Ans. 43,606.

What is the value of the permeability μ in the last case?

Ans. 1.86.

If the magnetic induction is 15,700 lines to the square centimeter and the intensity of the magnetic field producing it is 22 lines of force to the square centimeter, what is the permeability of the iron in which it is produced? *Ans.* 713.6

Calculate the value of μ for $H = 583$, and $B = 19{,}810$. *Ans.* 33.9.

What is the reluctivity of the iron of the last example? *Ans.* 0.0295.

What is the reluctivity corresponding to a permeability of 714?

Ans. 0.0014.

If a sample of iron has a permeability of 1,500, what field will be required to excite it to an intensity of magnetic induction of $\boldsymbol{B} = 15{,}000$?

Ans. 10 lines of force to the square centimeter.

A bar of iron is 112 cm. long and 75 sq. cm. cross-sectional area; its permeability at the given induction is 3.5. What are its permeance and reluctance? *Ans.* Permeance 2.34; reluctance 0.428.

A bar of iron is 44 inches long and 11 sq. inches cross-sectional area, and its permeability at the given inductance is 3.5. Calculate its permeance and reluctance. *Ans.* Permeance 2.22; reluctance 0.45.

What is the field in the center of a coil 19 cm. long, of 1,015 turns, with a current of 0.09 C.G.S. units?

Ans. 60.4 lines of force to the square centimeter.

What would be the effect of a current of 0.9 ampere in the same coil?

Ans. As 0.9 ampere = 0.09 C.G.S. unit, the result would be the same.

If the above coil was 2 cm. in internal diameter, what would be the total number of lines of force produced? *Ans.* 189.8 lines of force.

10 amperes are passed through a coil of 275 turns; a bar of iron 39 cm. long and 6 sq. cm. cross-sectional area is within it. Assuming that at the given induction $\mu = 40$, how many lines of force will pass through it? *Ans.* 21,266 lines of force.

A core of iron is 12 inches long and 3 sq. inches in section. Taking μ as 29 and the ampere turns as 750, what will be the magnetic induction $\boldsymbol{B}$? *Ans.* 17,346 lines of force.

What ampere turns are required to force 25,000 lines of force through a bar of iron (taking μ as being 2,500), 50 cm. long and 3.22 sq. cm. section? *Ans.* 123.5.

What ampere turns are needed to force 129,000 lines of force through an iron core (taking μ at 30) 100 in. long, 12.9 sq. in. section?

Ans. 10,443.

An electro-magnet is excited by a field of 4 lines to the square centimeter, causing 9,000 lines of force to pass through each square centimeter of the area of contact between the magnet poles and armature. The area of contact is 5 sq. cm. Calculate the traction.

Ans. 16,422 grams.

Let the area of contact be 0.775 sq. inch, the field be 25.80 lines of force to the square inch, producing an induction of 58,050 lines of force to the square inch. Calculate the traction in pounds.

Ans. 36.2 pounds.

A field of 15 lines of force to the square centimeter is produced and used to excite traction between a magnet and its armature. The area of contact is 0.5 sq. cm. and the traction is 4,841 grams. Calculate the permeability of the iron. *Ans.* 1,030.

The area of contact between the face of an electro-magnet and its armature is 1 square inch; the field is 2,272 lines to the square inch, producing a total induction through the area of contact of 19,820 lines to the square inch. Calculate the traction in pounds.

Ans. 5.346 pounds.

CHAPTER XV.

CAPACITY AND INDUCTANCE.

Capacity. — Measure of Capacity. — Capacity of Parallel Plates. — Equations for Capacity Calculations. — Energy of a Charge. — Specific Inductive Capacity. — Measure of Specific Inductive Capacity. — Relation of Absolute Potential, Capacity and Quantity. — Value of Absolute Elective Potential. — Potential Difference and Transfer of Quantity. — Heat Analogy of Potential and Capacity. — Inductance. — The Henry and Rate of Current Change. — Inductance Formulas. — Variation of Rate of Current Change. — Energy of the Electromagnetic Field of Force. — Problems.

Capacity. If a quantity of electricity be imparted to an insulated conductor it will change its potential. The capacity of a body is equal to the number of units of quantity which must be given it to change its potential one unit of potential. The potential is usually the potential difference between two conductors; theoretically the absolute potential may be considered.

Example. A condenser receives 11 units of electricity, causing a potential difference of 10^8 units. Calculate its capacity.

Solution. If 11 units of quantity change its potential 10^8 units, it would evidently require $\frac{11}{10^8} = 11 \times 10^{-8}$ units of quantity to change its potential one unit. This is the numerical value of its capacity. The change in potential will be that which is measured between its two sets of leaves.

Measure of Capacity. The capacity of a conductor or condenser is equal to the quantity of a charge divided by the potential which such charge will impart to it. It is numerically equal to the reciprocal of the potential which a unit charge will impart.

Calling quantity Q and capacity K and expressing potential difference by $V - V'$, we have

$$K = \frac{Q}{V - V^1}.$$

Example. A charge of one unit of quantity is imparted to a condenser. The potential difference between its plates is thereby raised to 12 units. What is its capacity?

Solution. It is the reciprocal of the potential, $\frac{1}{12}$ unit of capacity.

The unit of capacity is the farad; the micro-farad is generally used in practical computations. (Page 116.)

Capacity of Parallel Plates. The capacity of conductors of different forms generally has to be calculated by the higher mathematics. The capacity of two plates facing each other is thus determined if one or two assumptions are made.

Let the potential difference between the two plates be $V - V'$. Assume that the lines of force all proceed straight across from plate to plate and that they are evenly distributed. The force of attraction between the two plates is $4\pi\sigma$, in which σ is the surface density, so that $\sigma \times$ surface area is the total charge of the surface. The potential difference multiplied by the quantity gives the energy of the charge; therefore for unit potential the energy is numerically equal to the potential difference. The distance from surface to surface being t, the energy of the charge per unit of surface is $4\pi\sigma \times t$, and for unit potential this is equal to $V - V'$.

This gives $4\pi\sigma \times t = V - V'$ (1) and $\sigma = \dfrac{V - V'}{4\pi t}$ (2).

Let Q represent the charge on one of the plates whose surface area is S. Then the charge will be $S \times \sigma = \dfrac{V - V}{4\pi t} \times S = Q.$

The charge on one plate required to impart a potential difference of unity between them is the capacity of the condenser. This charge is given by the expression $\dfrac{Q}{V - V'} = K = \dfrac{S}{4\pi t}$. The capacity is directly proportional to the area and inversely proportional to the distance separating the plates. The nearer they are to each other the greater will the capacity be.

Example. Calculate the capacity of a circular plate 1 square meter (= 10,000 sq. cm.) in area, $\frac{1}{10}$ millimeter distant from another plate of identical size and shape.

Solution. Substituting in the formula gives

$$K = \frac{10,000}{4\,\pi \times 0.01} = 79,577 \text{ C.G.S. units} = 79,577 \times 10^{-6}$$

farads, or 79.6 microfarads. One square meter is equal to 10,000 sq. cm. and $\frac{1}{10}$ millimeter is equal to 0.01 cm.

Equations for Capacity Calculations. The charge of a condenser is equal to the capacity multiplied by the e.m.f. due to the charge. This gives the equation

$$\text{Charge} = \text{capacity} \times \text{e.m.f.} \qquad (1)$$

and by transposing,

$$\text{Capacity} = \frac{\text{charge}}{\text{e.m.f.}} \quad (2) \qquad \text{and e.m.f.} = \frac{\text{charge}}{\text{capacity}}, \quad (3)$$

which three equations serve for the solution of the simpler problems in capacity.

Example. A condenser has a capacity of 1.3 microfarads, and an e.m.f. of 1,325 volts is applied to its terminals. Calculate the charge it will receive.

Solution. By equation (1) the charge is equal to $0.000,001,3 \times 1,325 = 0.001,723$ coulomb.

Example. A charge of 0.000,931 coulomb is given a condenser, and an e.m.f. of 625 volts is produced between its terminals. Calculate the capacity of the condenser.

Solution. By equation (2) the capacity is $\frac{931}{625} \times 10^{-6} = 1.49 \times 10^{-6} = 1.49$ microfarad.

Example. What e.m.f. will a charge of 0.000,725 coulomb produce in a 2-microfarad condenser?

Solution. By equation (3) it will be $\frac{725}{2} = 362.5$ volts. The

decimals are omitted because each is of the same order; it is the division of microcoulombs by microfarads.

Example. A current of 75 amperes average intensity flows for $\frac{1}{200}$ second into a system of 39 microfarads capacity. Calculate the e.m.f. developed.

Solution. $75 \times \frac{1}{200} = 0.375$ coulomb. $0.375 \div 0.000{,}039 =$ 9,615 volts.

Energy of a Charge. A charged condenser is a seat of potential energy. When discharged it delivers electric quantity at a definite e.m.f., which constitutes electric energy. As we have seen, the charge of a condenser, which is the quantity of electricity it can discharge, is equal to the capacity multiplied by the e.m.f. This e.m.f. is the maximum e.m.f. due to the charge, and it is the initial e.m.f. of the discharge. The final e.m.f. of the discharge is 0, and as the diminution is uniform the average e.m.f. of the discharge is one-half the initial. Calling the initial e.m.f. e, the quantity discharged is capacity $\times$ e; the average e.m.f. of discharge is $e/2$; and the energy of discharge is the product of the two, or

$$\text{Energy of a charged condenser} = \tfrac{1}{2}\ \text{capacity} \times e^2.$$

Example. What is the energy in a 2.9-microfarad condenser charged to 375 volts?

Solution. Applying the equation we find $\frac{1}{2} \times 2.9 \times 10^{-6} \times 375^2 = 0.204$ joule.

By the doctrine of the conservation of energy the above equation gives the energy required to charge a condenser.

Specific Inductive Capacity. The capacity of two conductors facing each other and forming a condenser depends on their area, on the distance separating them, and on the material between them. This material must be a dielectric or non-conductor of electricity. The capacity of a condenser is proportional to a constant, which constant expresses a property of

the dielectric separating its surfaces. This property is called the specific inductive capacity of the dielectric. It is also called dielectric power, the dielectric constant, permittivity, and perviability.

Measure of Specific Inductive Capacity. The specific inductive capacity of air is taken as unity. That of any other dielectric is a number expressing the ratio of the capacity of an identical condenser with the other dielectric to that of one with air between its surfaces.

Example. The specific inductive capacity of sulphur is 3.2. If a certain air condenser has a capacity of 5 microfarads, what would the capacity of an identical condenser with sulphur between its surfaces be?

Solution. It would be $5 \times 3.2 = 16$ microfarads.

Example. Taking the specific inductive capacity of glass as 7, compare the capacities of sulphur and glass condensers, and calculate the capacity of a condenser which with sulphur between its plates has a capacity of 4 microfarads, when an equal thickness of glass is substituted for the sulphur.

Solution. The ratio is $3.2 : 7 = 1 : 2.19$ (nearly). For the condenser in which glass is substituted for sulphur we have the proportion

$$3.2 : 7 :: 4 : x = 8.75.$$

Or the capacity of a condenser which with sulphur was 4 microfarads would be increased to 8.75 microfarads by substituting glass for sulphur.

The utility of tables of dielectric constants is of limited value, as the values vary greatly with different observers.

Relation of Absolute Potential, Capacity, and Quantity. The absolute electrical potential at a point or place is a mathematical expression with a numerical value.

Potential expresses a relation between places or loci of force such that energy would be required to transfer a quantity from

one place to the other. A level plane, a table top for instance, is a locus of force, gravity attracting all objects on it. All parts are at the same potential because equally distant from the earth. Hence it takes no energy to move a weight from one part of the top to another, except for friction and inertia, because there is no change of potential. The floor is at a different potential because nearer the earth; hence to move a weight from one to the other involves energy. The criterion of the existence of a potential difference is the necessary change of energy relations in moving a quantity from one place of potential to another.

Value of Absolute Electric Potential. Absolute electric potential is numerically equal to the energy needed to bring a unit quantity of electricity from an infinite distance to the point where the potential is. This place might be any insulated conductor. Absolute electric potentials are calculated by the higher mathematics. They have little connection with practical electric work. Electric potential is equal to the quotient of quantity divided by capacity.

Calling potential V, the formula is

$$V \text{ (absolute) or } V - V' = \frac{\text{charge}}{\text{capacity}}.$$

Example. The capacity of a circular plate of radius r is $\frac{2r}{\pi}$. It receives a charge of 21 units of electric quantity. What is its potential, assuming the disk to be 30 cm. in diameter?

Solution. $2r$ = diameter = 30, and $\pi = 3.1416$. Substituting in the expression for capacity these values, the capacity of the disk is given as $\frac{30}{3.1416} = 9.55$ C.G.S. units. The quantity charged upon it divided by the capacity gives potential $= \frac{21}{9.55} =$ 2.19 C.G.S. units of electric potential.

Potential Difference and Transfer of Quantity. The difference of potential between any two conductors, or a differ-

ence of potential maintained between any two parts of a conductor, is an actual or possible cause of transfer of electric quantity. The transfer is the electric current, and the combination of quantity and potential implies and necessitates the expenditure of energy.

The numerical value of potential in any case is equal to that of the energy required to transfer a unit quantity of electricity against its action. In the case of absolute potential the quantity is supposed to be transferred from zero potential up to the potential stated, so that at the completion of the transfer the quantity will have the stated potential. In the case of potential difference the quantity is transferred from one to the other potential.

Example. 500 ergs are expended in transferring 29 units of electric quantity. What is the potential difference?

Solution. $500 \div 29 = 17.24$ units of potential.

Heat Analogy of Potential and Capacity. If we call temperature of a body its potential and call its specific heat its capacity, what has been said of electricity can be illustrated by the laws of heat. Thus the energy required to raise a body from a temperature m to a temperature n is equal to its mass or quantity multiplied by its specific heat or thermal capacity. The difference of temperatures is equal to the energy divided by the mass, exactly as in the last example.

Example. 500 ergs are expended in heating 29 grams of water. Taking the thermal capacity of water as 1, calculate the temperature or thermal potential which will be imparted to the water.

Solution. As potential difference is numerically equal to the energy required to raise unit quantity to the same potential or through the same potential difference, the temperature imparted to the water will be the quotient of $500 \div 29 = 17.24°$ C. This is only given as a sort of analogy.

Example. The thermal capacity of aluminum being taken as 0.215, the product of this by any weight will represent the electric capacity of a condenser. Let heat imparted to it be called its charge and temperature be called potential. If a charge of 11 heat units be imparted to 12 grams of aluminum, what potential will be developed?

Solution. The formula is $V - V' = \frac{\text{charge}}{\text{capacity}}$. Substituting for each quantity its value, taking $0.215 \times 12 = 2.58$ for capacity, we have

$$V - V' = \frac{11}{2.58} = 4.26° \text{ C.}$$

Inductance. The electro-magnetic field of force is a form of potential energy. It is produced by passing a current of electricity through a conductor. The quantity of energy varies according to the conditions and environment of the circuit. To create the field the expenditure of energy is required, while it is maintained without the expenditure of energy. Inductance expresses the conditions and environment of the circuit.

Suppose that a unit of potential, as a volt, is applied to a circuit possessing inductance. Any independent circuit has inductance; one surrounding an iron core, such as an electro-magnet coil, has high inductance. A current would be the result, but instead of at once taking the value according to Ohm's law it would grow in intensity and would build up a field of force. The ultimate value of the field of force would be that which the full current calculable by Ohm's law would create and maintain, and as soon as such field was produced the current would cease to increase but would continue at the value calculable by Ohm's law.

The energy of the current at the given e.m.f. would be expended in creating the field until the field attained the value due to the current which would be produced in the circuit according to Ohm's law.

The Henry and Rate of Current Change. The unit of inductance is the henry. It is the inductance of a circuit which requires the maintenance of one volt e.m.f. to increase the current passing through it at the rate of one ampere per second. Its symbol is L.

Example. To increase a current passing through a certain circuit from 0 ampere to 39 amperes in 0.001 second 117 volts are required. What is the inductance of the circuit?

Solution. The rate of change is $39 \div 0.001 = 39{,}000$ amperes per second. If 117 volts are required to maintain this rate of change, then to maintain a rate of change of 1 ampere would require $117 \div 39{,}000 = 0.003$ volt. The inductance of the circuit is 0.003 henry.

Example. The inductance of a circuit is 2 henrys. What e.m.f. must be employed to increase a current through it from 1 ampere to 24 amperes in $\frac{1}{2}$ second?

Solution. The rate of change is $(24 - 1) \div \dfrac{1}{2} = 46$ amperes per second. Multiplying the rate of change by the inductance of the circuit gives the e.m.f. required; $46 \times 2 = 92$ volts.

Inductance Formulas. The formulas involved in these calculations may be thus expressed:

$$\text{Inductance in henrys} = \frac{\text{e.m.f.}}{\text{rate of change}}, \qquad (1)$$

$$\text{e.m.f.} = \text{inductance} \times \text{rate of change}, \qquad (2)$$

and a third one may be given,

$$\text{Rate of change} = \frac{\text{e.m.f.}}{\text{inductance}}. \qquad (3)$$

Example. If 21 volts are expended in increasing the current through a circuit of 0.004 henry inductance, what is the rate of change of current?

Solution. By formula (3) it is $21 \div 0.004 = 5{,}250$ amperes per second.

Variation of Rate of Current Change. Inductance at the instant an e.m.f. begins to act upon an inductive circuit is the only opposing factor, because until a current is flowing resistance is without effect. The necessary condition of equilibrium is brought about by the creation of counter e.m.f. equal to the impressed, and this counter e.m.f., as we have seen, is equal to the product of the inductance by the rate of change of current intensity.

After a current has begun to flow a part of the e.m.f. equal to the product of this current by the resistance is expended on maintaining the current and this amount is the *RI* drop; the rest of the e.m.f. produces an increase of current whose rate of increase or of change is such as to produce a counter e.m.f. exactly equal to the residual e.m.f. Knowing the inductance and the residual e.m.f. the rate of change is given by formula (3).

Example. An inductance of 0.04 henry is connected to a lighting circuit of 110 volts. Calculate the rate of change at the instant of connection, then at the instant the current has grown to 20 amperes, and when it has grown to 25 amperes. The inductance is of 4 ohms resistance.

Solution. Applying formula (3) gives as the initial rate of change $\frac{110}{0.04} = 2,750$ amperes per second. When the current has attained a value of 20 amperes, the e.m.f. expended on the inductive coil due to its resistance is its *RI* drop, or $4 \times 20 = 80$ volts. This leaves $110 - 80 = 30$ volts free to act to increase the current and to be in equlibrium with the counter e.m.f. caused by such increase. In other words, the current can only increase at a rate which will produce counter e.m.f. equal to the e.m.f. producing the increase. Applying formula (3) as before, the rate of change at the 20-ampere point is $\frac{30}{0.04} = 750$ amperes per second. At the 25-ampere point the *RI* drop is $4 \times 25 = 100$

volts; the e.m.f. acting to increase the current is $110 - 100 = 10$ volts, and the rate of change is $\frac{10}{0.04} = 250$ amperes per second.

Energy of the Electro-magnetic Field of Force. The electro-magnetic field of force is a seat of energy. We have seen that the product of inductance by the rate of current change gives the initial counter e.m.f. of the field. The counter e.m.f. diminishes until the field is fully formed when it is 0. The average counter e.m.f is therefore $\frac{1}{2} L \frac{i}{t}$. This multiplied by the quantity, or it, the current multiplied by the time, gives the energy in electric or equivalent units. Thus we have

$$\text{Initial counter e.m.f.} = L\frac{i}{t}, \tag{1}$$

$$\text{Average counter e.m.f.} = \frac{1}{2} L \frac{i}{t} \tag{2}$$

average counter e.m.f. $\times$ quantity = joules, and multiplying (2) by it we have

$$\text{Energy} = \frac{1}{2} Li^2. \tag{3}$$

Example. What is the kinetic energy in a circuit of 0.000,031 henry through which a current of 29 amperes is maintained?

Solution. Applying formula (3) we have

$$\text{Energy} = \frac{1}{2} \times 31 \times 10^{-6} \times (29)^2 = 0.013,036 \text{ joule.}$$

PROBLEMS.

If a charge of 0.2 coulomb give a potential difference of 250 volts between the plates of a condenser, what is its capacity?

Ans. 800 microfarads.

Calculate the capacity of a pair of circular plates 39 cm. in diameter and 0.1 cm. apart. *Ans.* 951 C.G.S. units; 0.95 microfarads.

A charge of 321 microcoulombs gives a potential difference of 750 volts. What is the capacity of the condenser receiving it?

Ans. 0.43 microfarads.

What e.m.f. will 0.001,750 coulomb produce in a microfarad condenser? *Ans.* 1,750 volts.

If a 1.5-microfarad condenser is charged to 321 volts, what will the measure of the charge be? *Ans.* 0.000,481 coulomb.

1.11 amperes flow for $\frac{1}{75}$ second into a microfarad condenser. What e.m.f. will be developed? *Ans.* 14,800 volts.

What is the energy in the charged condenser of the last problem?

Ans. 109.52 joules.

Into the air spaces of an air condenser of 2 microfarads capacity paraffine (specific inductive capacity 1.993,6) is poured. What is the capacity of the new condenser thus produced?

Ans. 3.987,2 microfarads.

Compare the ratios of two condensers of identical dimensions, one with sulphur (specific inductive capacity 3.2) and the other with paraffine as the dielectric.

Ans. The sulphur condenser has 1.6 times the capacity of the other one.

If sulphur is substituted for paraffine in a condenser of 3.11 microfarads capacity, what will the capacity of the new condenser be?

Ans. 4.98 microfarads.

2,942,280 ergs are required to transfer 0.333 C.G.S. unit of electric quantity from one place to another. What is the difference of potential of the two places? *Ans.* 8,916,000 C.G.S. units; 0.089 volt.

A current of 3 amperes is to be produced in a circuit of 10 henrys inductance in 3 seconds. What e.m.f. will be needed? *Ans.* 10 volt.

An e.m.f. of 200 volts produces a current of 39 amperes by acting on a circuit for $\frac{1}{11}$ second. What is the inductance of the circuit?

Ans. 0.466 henry.

If an e.m.f. of 114 volts acts upon a circuit having an inductance of 0.57 henry, what will the rate of change be?

Ans. 200 amperes per second.

An inductance of 0.057 henry is present in a circuit of 3 ohms resistance; an e.m.f. of 112 volts is maintained on the circuit. Calculate the rate of change (*a*) when the e.m.f. begins to act upon the circuit; (*b*) when the current has attained a strength of 29 amperes; (*c*) when the current has attained a strength of 37 amperes.

Ans. (*a*) 1,965 amperes per second.
(*b*) 439 amperes per second.
(*c*) 17.5 amperes per second.

An inductance of 0.17 henrys is present in a circuit of 2.9 ohms resistance; 1,110 volts e.m.f. is maintained on the circuit. Calculate the rate of change (*a*) at the first application of the e.m.f.; (*b*) at the expiration of the time required for the current to grow to 37 amperes; (*c*) the same for 350 amperes.

Ans. (*a*) 6,529 amperes per second.
(*b*) 5,898 amperes per second.
(*c*) 559 amperes per second.

What is the energy in the field produced by a current of 3.75 amperes passing through a circuit of 0.000,7 henrys? *Ans.* 0.004,921,875 joule.

CHAPTER XVI.

HYSTERESIS AND FOUCAULT CURRENTS.

Hysteresis Loss. — Steinmetz's Hysteresis Formula and Table. — Steinmetz's Formula Based on Weight of Iron. — Foucault or Eddy Currents. —Formulas for Laminated Cores. — Formulas for Wire Cores. — Copper Loss in Transformers. — Efficiency of Transformers. — Ratio of Transformation in Transformers. —Problems.

Hysteresis Loss. Energy is required to create a field of force, but no energy is required to maintain it, and if it is caused or allowed to disappear it gives off energy equal to that expended on its formation. If the field of force includes a mass of iron in its volume or space occupied by it, then there will be a loss of useful energy, due to the fact that the iron tends to retain some magnetism, and that a small amount of energy is expended in its demagnetization which is converted into heat energy and operates to increase the temperature of the iron. In many cases a transformer would work at an efficiency of nearly 100 per cent if there were no hysteretic loss.

The loss due to hysteresis is stated in energy units per cycle of magnetization and demagnetization per unit volume of the iron per value of magnetic induction (B), or else in watts per unit of weight of iron per value of magnetic induction at a stated frequency (cycles per second). The loss may be determined experimentally for any given sample of iron, or may be calculated by a formula of the empirical class due to Steinmetz.

Steinmetz's Hysteresis Formula and Table. Let h designate the hysteresis loss in ergs per cubic centimeter of the iron per cycle, and let η (Greek letter eta) designate a constant depending on the quality of the iron. B denotes the magnetic induction. Then

$$h = \eta B^{1.6},$$

in which h is ergs of hysteresis loss.

The following table from Steinmetz gives values of η for different irons.

Very soft iron wire,	0.002	Soft annealed cast steel,	0.008
Very thin soft sheet iron,	0.0024	Soft machine steel,	0.0094
Thin good sheet iron,	0.003	Cast steel,	0.012
Thick sheet iron,	0.0033	Cast iron,	0.0162
Most ordinary sheet iron for transformer cores,	0.004 to 0.0045	Hardened cast steel,	0.025

Example. The iron in a transformer core is subjected to 8,000 lines of force per square centimeter at the extremes of the cycle. Calculate the loss by hysteresis per cycle.

Solution. Substituting in the formula we have

$$h = 0.0045 \times (8{,}000)^{1.6} \text{ ergs.}$$

The logarithm of 8,000 is 3.903,090. Multiplying it by 1.6 gives the logarithm of $(8{,}000)^{1.6}$, which is 6.244,944, the number corresponding to which is $17{,}577 \times 10^2$. This gives as the value of h

$$0.0045 \times 17{,}577 \times 10^2 = 7{,}910 \text{ ergs,}$$

the hysteretic loss in iron of that quality ($\eta = 0.0045$) per cycle per cubic centimeter with a maximum induction of 8,000.

Steinmetz's Formula based on Weight of Iron. Iron weighs about 7.7 grams per cubic centimeter and a pound is equal to 453.6 grams. There are therefore $453.6 \div 7.7 = 58.9$ cubic cm. in a pound of iron. Call the number of cycles per second f, and the formula becomes for one pound and for f cycles per second, in ergs per second,

$$h = 58.9\, f\eta \boldsymbol{B}^{1.6},$$

and as 10^7 ergs per second are a watt, the formula becomes for watts of hysteresis loss and for n pounds

$$W = 58.9 \times 10^{-7} f\eta \boldsymbol{B}^{1.6} \times n.$$

Example. A two-pole dynamo armature makes 15 revolutions per second. It weighs 27 pounds. The average number

of lines of force per square centimeter of cross-sectional area of field is 12,000. Let the iron have a hysteresis factor of 0.003. Calculate the rate of loss.

Solution. Substituting in the formula we have

$$W = 58.9 \times 10^{-7} \times 15 \times 0.003 \times 12{,}000^{1.6} \times 27,$$

which gives as solution $W = 24$ watts. This is the rate at which energy will be absorbed, expending itself in heating the armature core. The answer of the formula can be equally well read in joules per second.

Within reasonably high values of B this formula gives results not over 3 per cent different from those obtained by experiment.

Foucault or Eddy Currents. The name Foucault currents or eddy currents is used to designate currents produced in the cores or other masses of metal of electrical machinery by the variations in the magnetic induction to which they are subjected in the operation of the machines. These currents do no direct injury unless they become so intense as to overheat the metal so as to affect the insulation, but are indirectly harmful as absorbing energy and thereby making the operation of the machinery less efficient, so that they act with hysteresis to produce a waste of energy.

Formulas for Laminated Cores. When a core is built up of sheets of iron insulated from one another the following formula expresses the loss in watts or in joules per second. In it

f = frequency. W = watts absorbed per cubic cm.
c = specific conductivity. B = magnetic induction.
a = thickness of plates.

We then have

$$W = \frac{\pi^2 f^2 a^2 B^2 c}{4 \times 10^{16}}.$$

There is another factor in the formula which is omitted because it is practically equal to unity. The value of c will vary widely

for different irons. For such as are used in converters and in armatures a high value is 1.02×10^5; this is at the temperature of zero; for each degree increase of temperature a decrease of conductivity of 0.365 per cent may be allowed for. This is applied to the value of c in the formula before substituting.

The value of c in the formula is in the reciprocal of ohms per cubic centimeter; the value of a is in centimeters; the value of $\boldsymbol{B}$ is in lines of force per square centimeter.

Example. A core is built up of plates of iron 0.2 cm. thick and of the conductivity 1.02×10^5 at zero C. In operation it becomes heated to the temperature of 200 degrees C. The frequency of alternations is 12 per second. The induction is $10{,}000 = 10^4$ lines of force to the square centimeter. Calculate the loss due to eddy currents.

Solution. The conductivity at 20 degrees C. is $[1.02 - (1.02 \times 0.00365 \times 20)] \times 10^5 = (1.02 - 0.07) \times 10^5 = 0.95 \times 10^5$. Substituting this and the other values in the formula we obtain

$$W = \frac{\pi^2 \times (12)^2 \times (10^4)^2 \times (0.2)^2 \times 0.95 \times 10^5}{4 \times 10^{16}}$$

$$= \frac{9.87 \times 144 \times 10^8 \times 0.04 \times 0.95 \times 10^5}{4 \times 10^{16}}$$

$$= 9.87 \times 1.44 \times 0.95 \times 10^{-3} = 0.0135 \text{ watt.}$$

This is the rate of loss per cubic centimeter of the core. The insulation is not to be included in the volume.

Separating the formula thus:

$W = \dfrac{\pi^2 \times c}{4 \times 10^{16}} \times f^2 \boldsymbol{B}^2 a^2$, and assigning an average value to c, we may express the fraction, $\dfrac{\pi^2 \times c}{4 \times 10^{16}}$, as a constant. Calling this constant b, the formula becomes

$$W = bf^2 \boldsymbol{B}^2 a^2.$$

To show how b is deduced assume the conditions of conductivity of the last problem. We then have as the value of b,

$$b = \frac{9.87 \times 0.95 \times 10^5}{4 \times 10^{16}} = 2.344 \times 10^{-11}$$

and the formula becomes

$$W = 2.344 \times 10^{-11} \times 144 \times 10^8 \times 0.04 = 0.0135 \text{ as before.}$$

This is too high a value of b to represent ordinary practice. An accepted value is 1.6×10^{-11}. As the value of b varies with every sample of iron and with every change of temperature, it is evident that the formula based on a constant value of b must be an approximate one only. Unless the temperature and conductivity of the iron are accurately known, the original formula will also be an approximate one.

Formulas for Wire Cores. If a core is made of wire, the formula must include the diameter of the wire in place of the thickness of the plates. For wire cores the formula is the following:

$$W = \frac{\pi^2 f^2 d^2 B^2 c}{16 \times 10^{16}}.$$

If the constant is to be used, it is equal to $\frac{\pi^2 \times c}{16 \times 10^{16}}$, which is the same expression as already used for the value of b, except that 16 in the divisor replaces 4; therefore any value of b that is applicable to the formula for laminated cores becomes applicable to wire cores by dividing it by 4. Thus for a wire core, instead of 1.6×10^{-11} we would have to use 4×10^{-12}, which is one-fourth of the value of b for plate cores. Calling this constant for wire cores b' we have, as the formula embodying the constant b' for wire cores,

$$W = b' f^2 B^2 d^2.$$

Example. A core is made of iron wire insulated as usual. The frequency is 24, the induction is 12,000 lines of force to the square centimeter, the conductivity of the iron at zero

is 0.78×10^5, the temperature of operation is 36 degrees C., and the wire is 0.3 cm. diameter. Calculate the eddy currents loss per cubic centimeter.

Solution. This statement calls for the application of the original formula for wire cores. The value of c is deduced as before; it is $[0.78 - (0.78 \times 0.00365 \times 36)] \times 10^5 = 0.68 \times 10^5$. Substituting this and the other values of the problem in the formula we have

$$W = \frac{9.87 \times (24)^2 \times 0.09 \times (12 \times 10^3)^2 \times 0.68 \times 10^5}{16 \times 10^{16}}$$

$$= \frac{9.87 \times 576 \times 0.09 \times 144 \times 0.68 \times 10^{11}}{16 \times 10^{16}} = 0.0313.$$

This is the value of the loss due to eddy currents per cubic centimeter of iron, in rate units (watts) or in joules per second.

Example. Make the same calculation but using the constant of the problem before modified so as to apply to wire.

Solution. The constant is 1.6×10^{-11}. For it to apply to wire it must be divided by 4, giving 4×10^{-12}. Substituting in the formula $W = b'f^2B^2d^2$,

$$W = 4 \times 10^{-12} \times (24)^2 \times (12 \times 10^3)^2 \times 0.09 = 0.0299,$$

which differs from the other solution on account of the constant c having a value slightly different from that assigned it in the first treatment of the problem.

It has been claimed that in all these problems the induction $\boldsymbol{B}$ should be raised to the 1.6 power instead of to the square.

Copper Loss in Transformers. The sources of loss in transformers are hysteresis, eddy currents, and copper loss. The former two have already been treated; the copper loss is simply watts expended in the primary and secondary coils, which are equal to the product of current by e.m.f. absorbed by each of the coils or to the resistance of each of the coils multiplied by the square of the current.

Denoting the primary current and resistance by I_1 and R_1, and denoting the same for the secondary by the same letters with subscript 2, the copper loss is given by the following formula:

$$\text{Per cent of copper loss} = \frac{R_1(I_1)^2 + R_2(I_2)^2}{\text{input in watts}} \times 100.$$

The multiplication by 100 is necessary to give the per cent factor; without such multiplication the result is a decimal fraction, which can be used equally well.

Example. A primary circuit delivers 3,200 watts to a transformer. The resistance of its primary is 3 ohms, the resistance of its secondary is 300 ohms. What is the per cent of copper loss for a primary current of 1.7 amperes and a secondary current of 0.8 ampere.

Solution. Substituting in the formula gives

$$\text{Per cent of copper loss} = \frac{[3 \times (1.7)^2] + [300 \times (0.8)^2]}{3{,}200} \times 100$$

$$= \frac{200.67}{3{,}200} \times 100 = 6\tfrac{2}{25} \text{ per cent.}$$

Efficiency of Transformers. The efficiency of a transformer is equal to the output divided by the input. If to be in per cents the quotient is multiplied by 100. Otherwise it will be in simple decimal notation. The output is equal to the input diminished by the sum of the eddy current loss, the hysteresis loss, and the copper loss. The constituents of this expression are all in watts. The efficiency is

$$\text{Per cent of effi.} = \frac{\text{input} - (\text{ed.cur.loss} + \text{hys.loss} + \text{cop.loss})}{\text{input}} \times 100.$$

Example. What is the efficiency of a transformer in which the eddy current loss is 52 watts, the hysteresis loss is 47 watts, and the copper loss is 62 watts, at an input of 11 horse-power?

Solution. 11 horse-power $= 746 \times 11 = 8{,}206$ watts. Applying the formula we have

$$\text{Per cent of efficiency} = \frac{8{,}206 - (52 + 47 + 62)}{8{,}206} \times 100$$

$$= \frac{8{,}045}{8{,}206} \times 100 = 98.04 \text{ per cent.}$$

This is about the maximum efficiency attained in practice.

Ratio of Transformation in Transformers. The ratio of transformation in a transformer is the ratio of the impressed e.m.f. to the e.m.f. delivered to the secondary. The ratio is equal to the turns in the secondary coil divided by the turns in the primary coil. It can be expressed thus:

$$\text{e.m.f. output} = \text{e.m.f. of primary} \times \frac{\text{turns in secondary}}{\text{turns in primary}}.$$

Example. A transformer is actuated by a voltage of 1,500 volts. The primary coil has 1,000 turns, the secondary has 33 turns. Calculate the e.m.f. delivered.

Solution. By the formula it is equal to $1{,}500 \times \frac{33}{1{,}000} = 49.5$ volts.

Example. What must the ratio of primary to secondary be to transform 2,000 volts to 60?

Solution. It must be $2{,}000 : 60 = 33\frac{1}{3} : 1$.

PROBLEMS.

In a field of 9,500 lines of force to the square centimeter at the maximum, with a hysteretic constant of 0.0045, calculate the loss of energy per cycle. *Ans.* 10,413 ergs.

With a maximum of 65,000 lines of force to the square inch, and with a core of 0.0043 hysteretic constant, what is the loss per cubic centimeter per cycle? *Ans.* 10,935 ergs.

A core weighs 357 pounds; the polarity changes 2,500 times a minute; the constant of the core is 0.0024, with a maximum field of 114,000 lines to the square inch. Calculate the rate of loss.

Ans. 1,313.8 watts.

A core weighs 250 pounds; there are 16 cycles per second; the maximum field is 21,000 lines to the square centimeter; the hysteretic constant is 0.003. Calculate the rate of loss. *Ans.* 581.9 watts.

The plates of a laminated core are 0.3 cm. thick; the iron of the core has a specific resistance of 1.04×10^5 at $0°$ C., with a coefficient of 0.00365 increase in resistance per degree C.; in operation it is heated to $25°$ C.; the frequency is 17 per second; the maximum induction is 15,000 to the square centimeter. What is the loss per cubic centimeter due to eddy currents? *Ans.* 0.1359 watt.

With a value for b in the short formula of 1.6×10^{-11}, a maximum induction of 14,780 lines of force to the square centimeter, frequency 16 per second, and a thickness of laminations of 0.27 cm., calculate the loss per cubic centimeter. *Ans.* 0.06523 watt.

A core is made of iron wire of 0.1 cm. thickness and carries a maximum induction of 23,500 lines of force to the square centimeter; the frequency is 25 per second; the conductivity of the wire is 0.78×10^5 at $0°$ C.; the temperature of operation is $39°$ C. Calculate the loss per cubic centimeter. *Ans.* 0.01414 watt.

Apply the short formula, taking the thickness of the wire at 0.13 cm., the frequency at 25 per second, the maximum induction at 23,400 lines per square centimeter, and a value of 4×10^{-12} for b'. Calculate the loss per cubic centimeter. *Ans.* 0.2313 watt.

The primary coil of a transformer has 900 turns and receives a voltage of 1,200. How many turns must there be in a secondary coil to give 16 volts? *Ans.* 12 turns.

A primary circuit delivers 3,100 watts to a transformer the resistance of whose primary and secondary coils are 60 ohms and 5 ohms respectively. If there is a current of 0.4 ampere in the primary coil and 3 amperes in the secondary, calculate the per cent of copper loss.

Ans. $1\frac{118}{155}$ per cent.

The same transformer has a hysteresis loss of 32.4 watts and an eddy current loss of 38.2 watts. Calculate the per cent of efficiency.

Ans. $95\frac{149}{155}$ per cent, or 96 per cent nearly.

CHAPTER XVII.

ALTERNATING CURRENT.

Induction of Alternating E.M.F. — Alternating Current. — The Sine Curve. — Sine Functions. — Cycle. Frequency. — Value of Instantaneous E.M.F. and Current. — Average Value of Sine Functions. — Effective Values.— Form Factor. — Reactance of Inductance. — Rate of Change. — Deduction of Ohmic Value of Inductance Reactance. — Impedance of Inductance and Resistance. — Lag and Lead. — Lag of Current. — Deduction of Ohmic Value of Capacity Reactance. — Combined Impedance of Inductance and Capacity. — Lead of Current. — Lag or Lead due to both Reactances. — Impedance of Inductance, Capacity, and Resistance Combined. — Angle of Lead or Lag. — Power and Power Factor. — Power Factor for both Reactances Combined. — Angle of Lag and Rate of Change.

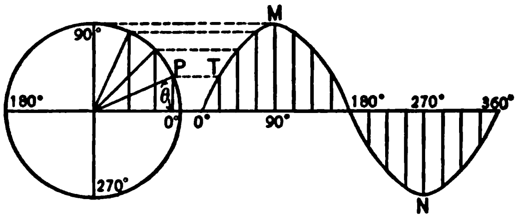
Induction of Alternating E.M.F. If a coil of wire rotates in a uniform field, the coil representing practically the circumference of a disk, electro-motive force will be generated in it. This e.m.f. will vary in amount and in polarity. Assume the coil at right angles to the field and turning. At that point no e.m.f. will be impressed upon it, because it cuts no lines of force. As it rotates it begins to cut lines at a rate increasing until a maximum rate is reached at the position when its plane is parallel with the lines of the field. Here the e.m.f. is at its maximum, and as it passes this position the e.m.f. begins to fall off, because the coil cuts the lines at a lower rate, until as it reaches the position at right angles to the lines of the field the e.m.f. falls again to zero value. So far the e.m.f. has risen and fallen, producing a field of one polarity, but as the coil in its rotation begins to move through the other half of its course the e.m.f. goes through exactly the same set of changes as before, except that the polarity of the field produced is the reverse of what it was before.

Alternating Current. If the ends of the coil are connected by a conductor a current will pass through it which will vary

in intensity by the same law regulating the e.m.f., and which will therefore vary from a maximum to zero, and which will go alternately in opposite directions as the polarity of the e.m.f. changes.

Such a current is called an alternating current, and the division of the science relating thereto is called alternating current electricity.

The Sine Curve. A current produced by the arrangement described is what is known as a sine current. If a horizontal line is drawn and is taken to represent the development of the circumference of a circle, called the generating circle, any given



length laid out or measured off on the line may be taken as representing an angular quantity, measured in radians or degrees. The entire line will represent 2π radians, or 360 degrees. Erect on the first half of the line a number of verticals each of which will be at a point referable to angular measurement, as at the 10-degree mark, 20-degree mark, and so on. For the second half of the line, as the polarity of the e.m.f. is supposed to reverse itself at the center of the line, other lines are erected, but are inverted in position, or directed downwards. Assume the radius of the generating circle to be 1. Each line is made of length equal to the sine of the angle represented by its position. The line on the 10-degree mark, for instance, would be 0.17365 high, and the lines on the 170-degree, 190-degree, and 350-degree marks would be of the

same lengths, because these are the sines of the angles named. The last two lines would be directed downwards. In this way any number of lines could be laid out and the relative lengths of the lines made to vary as the lengths of the sines of the angles to which the respective lines belong. The values of the sines can be taken from a table of natural sines. A line drawn through the ends of the perpendiculars takes the form of a wave; it represents a cycle or wave of current or of e.m.f. and is a sine curve.

With the exception of the values for the 90-degree and 270-degree points which are equal to the radius, because the sine of either of these angles is 1, the values of the lines, which are called ordinates of the curve, will be expressed as decimals of the radius, if the radius is taken as 1, for the basis of the decimals is the radius of the generating circle. This radius is equal to the quotient of the length of the horizontal line, which is called the abscissa of the curve, divided by 2π, which is 6.2832.

Example. A sine curve is to be laid out on a line 4.25 in. long. Where should the 10-degree ordinate be erected and what should its length be?

Solution. 10° is $\frac{10}{360}$ of the line. $4.25 \times \frac{10}{360} = 0.118$ in., which is the distance from the end at which the ordinate should be drawn. The radius of the generating circle is $\frac{4.25}{6.2832} = 0.677$. From a table of natural sines we find that the sine of 10 degrees is 0.17365. The length of the 10-degree ordinate is the product of the radius of the generating circle by this decimal, or $0.677 \times 0.17365 = 0.1175$ inch. As the sines of 10 degrees, 170 degrees, 190 degrees, and 350 degrees are equal, this is the length of the four ordinates at the four degree points stated.

Sine Functions. The vertical lines or ordinates erected on the base line as described, and whose lengths are proportional to

the values of current and e.m.f. at the angular distances indicated, are sines of the angles, consequently the currents and e.m.f.'s are called sine functions.

Cycle. Frequency. A complete cycle starting from zero increases to a maximum in one direction, returns to zero, increases to a maximum in the other direction, and returns to zero. If drawn as a sine curve it starts from the base line, rises to a crest and returns to the base line, and then accomplishes the equivalent below the base line. The number of times this action takes place in a second, or the number of cycles produced in a second, is the frequency of the circuit, of the current, or of the e.m.f.

Value of Instantaneous e.m.f. and Current. In the ordinary operation of alternating current systems the polarity of the e.m.f. reverses many times in a second. The e.m.f. and the resulting current, which reverses in direction exactly as the e.m.f. reverses in polarity, are indicated by two sine curves, often of different altitudes but necessarily of the same angular length. Under some conditions the two curves may coincide so as to be one. Knowing the value of the maximum e.m.f. of current, the value of the e.m.f. or current at any part of the cycle defined by its angular position can be stated in terms of E_{max} (E maximum), the maximum e.m.f. or I_{max} the maximum current. Calling the e.m.f. at any part of the cycle e, its value is given by the equation $e = E_{max} \sin \theta$, and the equivalent process gives the current as $i = I_{max} \sin \theta$, θ indicating the angular position of the sine function. Such are termed instantaneous values.

Example. An alternating current generator produces a maximum e.m.f. of 2,000 volts. What is the e.m.f. of the cycle when $\frac{3}{7}$ completed?

Solution. $360° \times \frac{3}{7} = 154° \; 17'$. This is the angle θ; its sine is 0.43392, and substituting in the formula,

$$e = 2{,}000 \times 0.43392 = 867.84 \text{ volts.}$$

Average Value of Sine Functions. The average value of the e.m.f. or of the current of an alternating current system may be calculated approximately by adding together a number of the values of evenly distributed sines proportional to the e.m.f.'s or currents for every five or ten degrees of the wave length, dividing by the number taken, and multiplying the result by the maximum e.m.f. or current. It can be done with one-quarter of a wave, which is 90 degrees.

Example. Calculate the average value of the e.m.f. of a sine wave whose maximum e.m.f. is 2,250.

Solution. From a table of natural sines we obtain the following values of the sines of 0 degrees, 10 degrees, 20 degrees, and so on up to 90 degrees: 0., 0.17365, 0.34202, 0.50000, 0.64279, 0.76604, 0.86603, 0.93969, 0.98481, 1.0000. Adding these together and dividing by 10, the number of values, we have $6.21503 \div 10 = 0.6215$.

Approximate average value of the sines $= 0.6215$ and multiplying by the maximum e.m.f. gives $2,250 \times 0.6215 = 1,398.4$ volts.

The result is only approximate, the true average being 1,432 volts. The result could have been made more accurate by taking more sine values, as one for every 5 degrees or for every 2 degrees.

The length of one-half of the wave length is πE_{max} in which E_{max} is the length of the radius of the generating circle. If we know the area of the space included between the horizontal base line and the half of the sine curve lying above or below it, it is evident that the quotient of the area divided by the length of the half of the base line in question will be the average length of the vertical lines representing the sine functions. By calculus the area of the half of the sine curve is determined; it is $2\, E^2_{max}$. For the value of the average e.m.f. this is to be divided by the length of the base line of the half of the curve, the value of which length is given above. This gives

$$\text{Average e.m.f.} = \frac{2\, E^2_{max}}{\pi\, E_{max}} = \frac{2}{\pi}\, E_{max} = 0.6366\, E_{max}.$$

Example. What is the average value of the e.m.f. of an alternating current system whose maximum e.m.f. is 1,200 volts?

Solution. It is $1{,}200 \times 0.6366 = 764$ volts.

If $e = \frac{2}{\pi} E_{max}$, as above, then $E_{max} = \frac{\pi}{2} e = 1.57\, e$. By this formula if e is given E_{max} can be calculated.

Example. The average value of the e.m.f. of an alternating circuit being 875, what is the maximum e.m.f.?

Solution. It is $875 \times 1.57 = 1{,}373.75$ volts.

Effective Values. With a constant resistance the energy of an active circuit varies with the square of the electro-motive force, the expression for energy being E^2/R, and also with the square of the current, the expression for energy being RI^2. If one of these is true the other must also be true, because by Ohm's law the current varies with the e.m.f. Either expression reduces to EI, or, in practical units, to watts, the unit of energy rate. It follows that an alternating current or an alternating e.m.f. will represent at any instant a heating or energy effect proportional to the square of its value at that instant. The average effect will vary with the average of these squares, and this average will be the value of a direct current or direct e.m.f. of the same energy effect. This energy relation of alternating currents and e.m.f.'s is their virtual or effective relation, and for an alternating circuit to have the same energy as a direct circuit of the same elements the square of its effective current or e.m.f. must equal that of the same element of the direct circuit. The relation of the currents or e.m.f.'s unsquared, then, is that of the square root of these squares. For the direct circuit this gives the original current and e.m.f.; for the alternating circuit it gives the square root of the average square. If the square root of the average square of an alternating current is equal to a direct current, the energy developed

in the passage of either one through the same resistance will be the same. The same applies to the expenditure of e.m.f.

Effective values of current or e.m.f. are indicated thus: I_{ef} or E_{ef}.

The square root of the average square constitutes the effective or virtual current or e.m.f., I_{ef} or E_{ef}. Its value is thus determined: A curve is laid off similar to a sine curve and on the same base linc, but the verticals or ordinates, instead of varying with the sines, are laid off equal to the squares of the sines. The area of the space thus inclosed is determined by calculus; it is equal to $\frac{\pi E^3_{max}}{2}$. If this area is divided as before by the length of the base line, πE_{max}, it will obviously give a quotient which is the average of the squares of the sines; this quotient is $\frac{E^2_{max}}{2}$, and its square root is the square root of the average squares and is equal to $\frac{E_{max}}{\sqrt{2}} = E_{ef}$, whence $E_{max} = \sqrt{2}\, E_{ef}$. The square root of 2 is 1.414; $\frac{1}{\sqrt{2}}$ is 0.707; so $E_{ef} = 0.707\, E_{max}$, and $E_{max} = 1.414\, E_{ef}$. The same formulas apply to the effective current values.

Example. What is the effective value of an alternating current whose greatest value is 31 amperes?

Solution. It is $0.707 \times 31 = 21.9$ amperes.

The relation of the maximum current or e.m.f. to the effective may be thus proved by trigonometry.

The law of the sine curve applied to current strength gives the equation, letting I stand for any instantaneous value of current, in this case at the point θ,

$$I = I_{max} \sin \theta, \qquad (1)$$

and squaring,

$$I^2 = I^2_{max} \sin^2 \theta. \qquad (2)$$

The sum of the squares of the sine and cosine of any angle is equal to 1. The sines and cosines vary in value by exactly the same law, so for the average functions we have

$$\text{Average } \sin^2\theta + \text{average } \cos^2\theta = 1, \tag{3}$$

and as the average sine is equal to the average cosine, we can substitute one for the other, and doing this we have

$$2 \text{ average } \sin^2\theta = 1 \tag{4}$$

and

$$\text{Average } \sin^2\theta = \frac{1}{2} \tag{5}$$

and

$$\sqrt{\text{average } \sin^2\theta} = \sqrt{\frac{1}{2}} = 0.707. \tag{6}$$

If in equation (1) we substitute the square root of average $\sin^2\theta$ for $\sin\theta$, the second member of the equation will be the average value of the effective current, because the result will be the square root of the average values of the squares of the currents, thus:

$$I_{ef} = 0.707\ I_{max}. \tag{7}$$

Form Factor. The quotient of the effective value of an alternating current or e.m.f. divided by the average value is the form factor. For the sine curve current this is equal to $\frac{0.707}{0.636} = 1.11$.

As alternating systems are generally operated on sine curves the form factor in practice is taken as 1.11.

Reactance of Inductance. When a current varies in strength it causes a change in the electro-magnetic field of force which always surrounds a current. This change is one of strength of field. As the current increases it builds up the field, producing more lines of force; as the current decreases the field diminishes in strength also, the lines disappearing. Thus if a current is normally an increasing one its increase will be opposed by the

above action, because energy has to be expended to build up a field. If it tends to decrease, the tendency to decrease will be opposed also, because energy is given off in the reduction of a field, and this action tends to increase a current in the original direction or to diminish the rate of decrease of a decreasing current which produced or maintained the field.

The effect of inductance as outlined above is then to oppose the normal changes or normal action of an alternating current. This effect is called reactance. Its value depends on the rate of change of the current and on the inductance of the circuit. To put the effect of inductance into a form available for calculation, it can be expressed as equivalent to a definite number of ohms.

Rate of Change. The rate of change of an alternating current at any point expressed in degrees is equal to the product of the maximum current by the frequency by the cosine of the angle of position θ by 2π. The equation is

$$\text{Rate of change} = 2\,\pi f I_{\max} \cos\theta.$$

The numerical value of the rate of change is independent of its positive or negative sign, so that the sign of $\cos\theta$ is disregarded.

Example. What will be the rate of change in an alternating current of 133 frequency, effective value 65 amperes, at 180 degrees?

Solution. The maximum current is equal to $65 \times 1.414 =$ 92 amperes. The cos 180 degrees is -1. Substituting in the formula and disregarding the sign we have

$$\text{Rate of change} = 92 \times 2\,\pi \times 133 = 76{,}881 \text{ amperes.}$$

Example. In a sine current where is the maximum rate of change?

Solution. The variable in the equation is $\cos\theta$. The points where this has its highest values are the points of maximum rate of change. From trigonometry we know that the cosines of

0 degrees, 180 degrees, and 360 degrees are the highest in value; therefore the rate of change is highest at these points. The minimum rate is at the 90-degree and 270-degree points, where the cosines are of zero value, and consequently the rate of change at these points is 0.

Deduction of Ohmic Value of Inductance Reactance. The rate of change of the current at any point θ is given by the expression $2\ \pi f I_{max} \cos\ \theta$. The period of greatest rate of change is that at which $\cos\ \theta$ has the greatest value, and the maximum value of a cosine is when the arc has a value of 0 degrees or of 180 degrees; its value is then 1. The e.m.f. due to inductance is equal to the product of the rate of change by the inductance. Calling the inductance L, the e.m.f. due to it is equal to $2\ \pi f L I_{max}$ at the point of maximum value. By Ohm's law the e.m.f. is equal to RI_{max} for a current I_{max}, and we have $2\ \pi f L I_{max} = RI_{max}$, whence $R = 2\ \pi f L$. Therefore the ohmic equivalent of the inductance of an alternating circuit is equal to $2\ \pi f L$.

This gives the formula

$$\text{Reactance} = 2\ \pi f L, \tag{1}$$

in which L is the inductance of the circuit in henrys, and f is the frequency of the current. The value is given in equivalent ohms.

Example. A coil of wire is of such inductance that a current changing at the rate of one ampere a second induces a counter e.m.f. of 0.025 volt. An alternating current changing 100 times a second passes through it. Omitting any consideration of the true ohmic resistance, or assuming that it is so small as to be negligible, what is the ohmic equivalent of the reactance?

Solution. From the data of the problem, $L = 0.25$, $f = 100$, and we have

$$\text{Reactance} = 2\ \pi \times 100 \times 0.025 = 15.7 \text{ ohms (equivalent).}$$

The frequency of a current is the number of periods or waves per second in its operation. If T is the time of a period, then the frequency of the current is obtained by dividing 1 second by the time of a period; or $F = f = \frac{1}{T}$, which expression introduced into the reactance formula, as given above, makes it read

$$\text{Reactance} = \frac{2\pi L}{T}. \tag{2}$$

The formulas are exactly equivalent, and some authors use one and some the other.

Example. In a circuit of 0.051 henry an alternating current is produced with a period value of $\frac{1}{175}$ second. Calculate the reactance.

Solution. Substituting in the last formula,

$$\text{Reactance} = \frac{2\pi L}{T} = 56 \text{ ohms (equivalent).}$$

Angular velocity is equal to angle traversed in a second. If expressed in radians, and if T is the time required to traverse an angle of the value 2π, then such angular velocity has the value of $\frac{2\pi}{T}$. Angular velocity being denoted by ω (Greek letter omega) we have

$$\omega = \frac{2\pi}{T}, \tag{3}$$

and substituting this in the equation, Reactance $= \frac{2\pi L}{T}$ gives a third form of the reactance equation which is often used,

$$\text{Reactance} = L\omega. \tag{3}$$

Example. A current has a frequency of 133 and is passing through a circuit of 0.090 henry. Calculate the reactance by the angular velocity formula.

Solution. The time required for the completion of a period or wave is $\frac{1}{133}$ second, and the angular length of a period is

2π, or 6.2832 radians. The angular velocity of the current is therefore $6.2832 \div \frac{1}{133} = 835.66$ radians. Substituting this value in the reactance formula we have

$$\text{Reactance} = 0.090 \times 835.66 = 75.21 \text{ ohms (equivalent).}$$

For general purposes the formula (1), using frequency directly, is the most convenient. 'Equivalent' is often omitted, but is then to be understood.

Impedance of Inductance and Resistance. Reactance and resistance act together in an alternating current circuit to reduce a current due to a given e.m.f. Their combined action is equal to the square root of the sum of their squares, and their combined action is called impedance. Using the value of reactance of equation (1) we have

$$\text{Impedance} = \sqrt{R^2 + (2\pi f L)^2},$$

and equations (2) and (3) would give for the same the values

$$\sqrt{R^2 + \left(\frac{2\pi L}{T}\right)^2} \quad \text{and} \quad \sqrt{R^2 + (L\omega)^2}.$$

Example. A coil of wire has a resistance of 23 ohms and an inductance of 0.021 henry. What is its impedance for a current of frequency 110?

Solution. $2\pi f L = 6.2832 \times 110 \times 0.021 = 14.5.$ Then

$$\sqrt{(14.5)^2 + (23)^2} = \sqrt{739.25} = 27.2 \text{ ohms},$$

which is the impedance of the circuit for a current of the given frequency.

Lag and Lead. If the ends of the coil we have described are connected, the current due to the e.m.f. impressed on it will take the form of waves of exactly the angular length of the e.m.f. waves and of the same form. The current waves may correspond in position with the e.m.f. waves — crest lying over crest and both sets of waves crossing the base line at the same point —

or the two sets may vary in position. One set may reach the crest before the other does; the one in arrears is then said to lag. The amount of its lag is measured on the base line and is referred to the angular measurement of this line or, what is the same thing, to that of the generating circle. The lag is stated in degrees generally, sometimes in radian measurement. The set of waves in advance of the other is said to lead, and its position is stated, in angular measurement also, as the angle of lead.

If the height of the waves is used to indicate the measurement of current or of e.m.f., the two sets may vary in height. As a matter of convenience and to distinguish the two sets from one another they are often drawn of different heights.

Example. A set of current waves cross the base line $\frac{1}{10}$ of its length behind or later than the e.m.f. waves. What is the angle of lag?

Solution. The length of the line in angular measurement is 360 degrees, and $360 \times \frac{1}{10} = 36$ degrees, which is the angle of lag. To express it in radian measurement multiply the length of the base line in radians, 2π or 6.2832 radians, by $\frac{1}{10}$, giving 0.62832 radian as the angle of lag. As a radian is equal to 57.3 degrees, as a test of the correctness of the operations we may multiply it by 0.62832 and see if it gives the first result, 36 degrees. $57.3 \times 0.62832 = 36.002$ degrees, which is well within the limits of the decimals used.

Lag of Current. This is produced by inductance; the angle of lag, indicated by ϕ (Greek letter phi), is the angle whose tangent is equal to the quotient of the reactance of induction divided by the resistance. This gives the equation

$$\text{Tangent of angle of lag} = \tan\phi = \frac{\text{reactance}}{\text{resistance}} = \frac{2\pi f L}{R}$$
$$= \frac{2\pi L}{RT} = \frac{L\omega}{R}.$$

Example. A circuit through which a current is passing has a resistance of 3 ohms, and with the current in question going through it has a reactance of 5 ohms. What is the lag of current?

Solution. Tan $\phi = \frac{5}{3} = 1.666$, whence from a table of natural tangents or logarithmic functions we find $\phi = 59° 2' 6''$.

Example. A circuit has a resistance of 2.3 ohms and an inductance of 0.0034 henry. An alternating current with a frequency of 125 passes through it. Calculate the lag.

Solution. The reactance is $6.2832 \times 125 \times 0.0034 = 2.67$. $2.67 \div 2.3 = 1.16$. This is the tangent of the angle of lag corresponding to the angle $49° 14' 9''$.

Deduction of Ohmic Value of Capacity Reactance. If an alternating circuit is opened and has no capacity, no current can be produced in it. If capacity is present, then an alternating current will be produced by alternating e.m.f. The action of the capacity referred to the current wave is the following: As the wave starts from zero value and rises to its maximum value, the current is due to the discharge of the capacity, which would be represented by a condenser. In the case of a sine current the period required for the current to pass from zero value to maximum value is one-quarter of a cycle. At the beginning of the cycle the condenser is charged to the maximum amount it receives in the operation of the circuit. At the end of the quarter cycle, when the current is of maximum value, the condenser is completely discharged. The condenser now begins to receive a charge, and continues to receive it during the next quarter of a cycle, its charge attaining its maximum value when the current is of zero intensity.

It follows from the above that the maximum charge of a condenser in an alternating circuit is equal to the average value of the current multiplied by the time of charge, which is one-quarter of a cycle. If practical units are used the result will be

given in coulombs. As the period of a cycle is the quotient of 1 divided by the frequency, the quarter of a cycle is $\frac{1}{4f}$, and the value of the charge at the end of the quarter cycle is $I_{av} \times \frac{1}{4f}$. The e.m.f. of a condenser is equal to the quotient of the charge divided by the capacity. Calling the capacity of a circuit K we have as the value of the e.m.f. due to the capacity

$$\left(I_{av} \times \frac{1}{4f}\right) \div K = \frac{I_{av}}{4fK}.$$

But $I_{av} = I_{max} \times \frac{2}{\pi}$, or $\frac{2 I_{max}}{\pi}$, and substituting this value of I_{av} in last expression gives as the value of the e.m.f. due to capacity at the point of maximum value, which e.m.f. is opposed to the impressed e.m.f. and therefore is counter e.m.f.,

$$\text{Maximum counter e.m.f. of capacity} = \frac{I_{max}}{2\pi fK}.$$

By Ohm's law e.m.f. $= RI$, therefore as $\frac{I_{max}}{2\pi fK} = I_{max} \times \frac{1}{2\pi fK}$, it follows that $\frac{1}{2\pi fK}$ is the ohmic equivalent of the capacity reactance, or virtually expresses the resistance equivalent of capacity. This gives the formula

$$\text{Reactance of capacity} = \frac{1}{2\pi fK}.$$

Example. If a 35-microfard capacity is introduced in a circuit of 125 frequency, what will its reactance be?

Solution. Substituting in the formula we find

$$\text{Reactance} = \frac{1}{2\pi \times 125 \times 0.000{,}035} = 36.4 \text{ ohms.}$$

Combined Impedance of Inductance and of Capacity. It will be seen from what has been explained in the last few

lines that the two reactances work in opposition to each other in the sense that the reactance of induction acts in direct proportion to the quantity $2\pi fL$ and the reactance of capacity in inverse proportion to the quantity $2\pi fK$. The net reactance due to both, when both are present in a circuit, is obtained by subtracting one from the other.

Example. A current has a frequency of 150. It passes through a circuit of 22 microfarads capacity and of 0.015 henry (15 millihenrys) inductance. Calculate the reactance of the two.

Solution. The inductance reactance is $2\pi \times 150 \times 0.015 =$ 14.14 ohms.

The capacity reactance is $\dfrac{1}{2\pi \times 150 \times 0.000{,}022} = 48.23$ ohms.

The total reactance of the circuit is $48.23 - 14.14 = 34.09$ ohms.

Lead of Current. Capacity acts as regards lead and lag in the reverse sense of inductance; it causes a lead of the current, the tangent of the angle of lead being given by the quotient of its reactance divided by the resistance of the circuit. The tangent is given a negative sign because lead is opposed to lag and because the positive value is assigned to lag. The formula is

$$\text{Tangent of angle of lead} = \tan\phi = -\frac{\text{reactance}}{\text{resistance}} = -\frac{\frac{1}{2\pi fK}}{R}.$$

Lag or Lead due to both Reactances. Where both inductance and capacity are present the tangent of the angle of lag or of lead as the case may be is the algebraic sum of the two reactances divided by resistance. If the sign is positive it is an angle of lag; if the sign is negative it is an angle of lead.

The giving of the positive and negative signs as described and the application of algebraic addition distinguishes between lead and lag. Otherwise subtraction can be used if attention is given to whether lead or lag preponderates.

Impedance of Inductance, Capacity, and Resistance combined. When the three qualities of inductance, capacity, and resistance are present in a circuit the impedance is equal to the square root of the sum of the resistance squared plus the reactance squared. The reactance in this case is the algebraic sum of the two reactances, as just described.

The equation for impedance when inductance, capacity, and resistance are present in a circuit is

$$\text{Impedance} = \sqrt{R^2 + \left(2\,\pi f L - \frac{1}{2\,\pi f K}\right)^2}.$$

If reactance is due to capacity alone it has a negative sign, and if due to both inductance and reactance the sign will be negative if the capacity reactance is larger than the inductance reactance. But as the reactance is squared in the expression of resistance and as the square of a negative quantity has a positive sign, both reactances in all cases go to increase impedance.

Example. Calculate the impedance of a circuit carrying a 150 frequency current, with a circuit resistance of 23 ohms, inductance of 0.041 henry, and capacity of 51 microfarads.

Solution. The reactance of inductance is $2\pi \times 150 \times 0.041 = 38.64$ ohms. The reactance of capacity is

$$\frac{1}{2\,\pi \times 150 \times 0.000{,}051} = 20.8 \text{ ohms.}$$

The impedance is $\sqrt{(23)^2 + (38.64 - 20.8)^2} = 29.11$ ohms.

Angle of Lead or Lag. The tangent of the angle of lead or of lag is equal to the algebraic sum of the reactances divided by the resistance. The reactance of capacity is to be given a negative sign. The equation is

$$\text{Tangent of angle of lag or of lead} = \tan\phi = \frac{2\,\pi f L - \dfrac{1}{2\,\pi f K}}{R}.$$

If the tangent is negative the angle is an angle of lead; if positive it is an angle of lag.

Example. Calculate the angle of lag or lead in a circuit of 18 ohms resistance, 0.027 henry and 47 microfarads, with a current of 133 frequency.

Solution. We have

$$\operatorname{Tan}\phi = \frac{2\,\pi \times 133 \times 0.027 - \dfrac{1}{2\pi \times 133 \times 0.000{,}047}}{18}$$

$$= \frac{22.56 - 25.46}{18} = -\ 0.161.$$

From a table of natural tangents we find $\phi = 9°\ 9'$, the angle of lead, because its tangent has a negative sign.

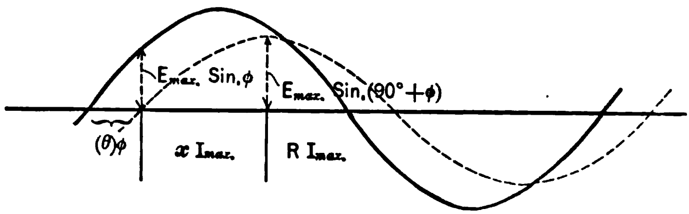
Proof of the Law of the Angle of Lag and Lead. — If an alternating e.m.f. is impressed upon a circuit, an alternating current will be produced whose frequency and form will be those of the impressed e.m.f.

At any given instant of time the value of e.m.f. required to produce the current existing at that instant will depend upon the resistance, inductance, and capacity of the circuit. The algebraic sum of the ohmic equivalent of inductance and the ohmic equivalent of capacity, taking the latter as of negative sign, is the ohmic value of reactance. It follows that the value of the current due to a given e.m.f. is determined by the two elements, resistance and reactance.

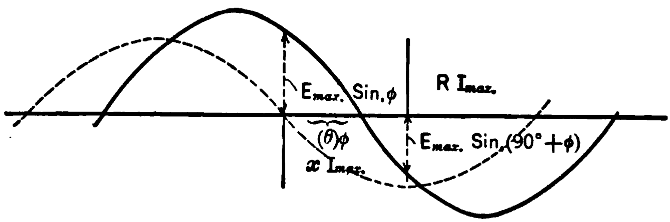
From Ohm's law we have $E = RI$. The value of I or of current in a half cycle varies from a maximum $I_{\mathbf{max}}$ to zero. In the absence of reactance the e.m.f. required to produce the current existing at any instant is equal to the product of the resistance of the circuit by such instantaneous value of the current or $RI_{\mathbf{inst}}$.

As reactance is due to variation in strength of current, it follows that at any point where the current is of constant value there is no reactance. In an alternating-current wave such

point coincides with the maximum current value, which is at the summit of the sine curve. Here the tangent to such curve is parallel to the base line and the current for an instant is of constant value, which value is, of course, I_{max}. The e.m.f. required to produce such current depends entirely on the resistance of the circuit, and is expressed by RI_{max}. This value is the height of the ordinate of the e.m.f. curve at the point where the current is of maximum value. As the e.m.f. curve is a sine curve, the value of the ordinate in terms of E_{max} is E_{max} sin θ.



The rate of change in an alternating current is greatest where its sine curve crosses the base line, at which point the current is zero value, and which point is 90 degrees removed from the summit of the curve.



The value of the impressed e.m.f. at this point depends entirely upon the reactance of the circuit. Resistance does not affect the value, because the current is of zero value. Calling the reactance x, we know that if we multiply it by the maximum value of current we have the counter e.m.f. due to reactance. Therefore the e.m.f. required to produce an alternating current

at the point of zero value of current where resistance plays absolutely no part is equal to xI_{max}. The expressions RI_{max} and xI_{max} are the values of the two ordinates of the sine curve of the impressed e.m.f. They are proportional to sines of angles, which angles differ in value by 90 degrees; in other words, they are proportional to the sines of complementary angles, and by trigonometry they are the sine and cosine respectively of such angles to the radius E_{max}. In terms of E_{max} these expressions become $RI_{max} = E_{max} \sin \theta$ and $xI_{max} = E_{max} \sin (\theta + 90°) = E_{max} \cos \theta$.

To obtain the absolute value of one of the angles, the value of the ordinate erected at such angle is divided by the value of the ordinate erected on the other angle; the result by trigonomtery is the tangent of the angle in question. The angle θ now becomes the angle of lead or lag; if we call it ϕ we have

$$\frac{xI_{max}}{RI_{max}} = \frac{x}{R} = \frac{E_{max} \sin \phi}{E_{max} \cos \phi} = \tan \phi,$$

or reactance divided by resistance gives the tangent of the angle of lag or of lead. If the ohmic value of the capacity reactance exceeds that of the inductance reactance, giving a negative sign to $\tan \phi$, the angle is an angle of lead and *vice versa*.

Power and Power Factor. The power of an alternating system is equal to the product of the effective e.m.f. and effective current, provided there is neither lag nor lead. In this case the product of the e.m.f. and current value is always the product of two negative or of two positive quantities; hence it always has a positive sign. But when there is lag or lead there is sometimes a product of positive e.m.f. by negative current, sometimes of negative e.m.f. by positive current; both of these products are negative. Then there are products of current by e.m.f. where both are of the same sign, which products are positive. The algebraic sum of the products for all angular values gives the

power, which in the case of a lag is evidently less than when there is no lag. The power of an alternating current system is equal to the product of the effective current by the effective e.m.f. multiplied by the cosine of the angle of lag. The cosine of the angle of lag is called the power factor.

Example. Assume an angle of lag of 49° 14′ 9″ and calculate the power, the effective current being 34 amperes and the effective e.m.f. 150 volts.

Solution. The cosine of the angle of lag 49° 14′ 9″ is 0.65295. The product of the effective e.m.f. by the effective current by the power factor gives the power; it is $150 \times 34 \times 0.65295 =$ 3,330 watts. If there were no lag or lead the power would be $150 \times 34 = 5{,}100$ watts.

Example. What is the power in a circuit with a lag of 90 degrees?

Solution. As the cosine of 90 degrees is zero, the power of such a circuit is zero.

The power factor is thus deduced by trigonometry:

Let the angle of period or of position of an alternating e.m.f. be designated by θ and the angle of lag of the current by ϕ. Then the angle of period of the current will be $\theta - \phi$. Then the values of e.m.f. and of current will be at the time of this period

$$E = E_{max} \sin \theta \qquad (1)$$

and

$$I = I_{max} \sin (\theta - \phi). \qquad (2)$$

If these equations are multiplied together the products will be the power of the circuit at that time. This gives

$$EI = E_{max} I_{max} \sin \theta \sin (\theta - \phi). \qquad (3)$$

By trigonometry $\sin (\theta - \phi) = \sin \theta \cos \phi - \sin \phi \cos \theta$, and substituting in (3) gives

$$EI = E_{max} I_{max} \sin^2\theta \cos \phi - E_{max} I_{max} \sin \theta \cos \theta \sin \phi. \qquad (4)$$

The angle ϕ is invariable, but for functions of θ the average

values must be substituted. The average value of $\sin^2\theta$ is $\frac{1}{2}$, in accordance with what has been proved on page 227. The average value of $\sin\theta\cos\theta$ is 0, because it has an equal sum of positive and of negative values whose average is 0. Substituting these values in equation (4) we have

$$\text{Average } EI = \frac{E_{\max} I_{\max}}{2} \cos\phi, \qquad (5)$$

the last term reducing to 0 because one of its factors, $\sin\theta\cos\theta$, is equal to 0.

We have found that $E_{ef} = \frac{E_{\max}}{\sqrt{2}}$ and $I_{ef} = \frac{I_{\max}}{\sqrt{2}}$, and substituting these values in equation (5) we have

$$\text{Average power (average } EI) = E_{ef}\, I_{ef} \cos\phi. \qquad (6)$$

Power Factor for both Reactances Combined. The power factor applies to capacity reactance exactly as it does to inductance reaction, and consequently applies also to the combination of the two reactances. From the practical standpoint the angles of lag and of lead are always treated as if they lay in the first quadrant of the circle. Even the negative sign of the tangent ϕ when it occurs is simply used to determine whether the angle is one of lag or of lead, but in finding the value of the angle from a table it is treated as a positive quantity. The power factor, which is a cosine of an angle between 0 degrees and 90 degrees, must always have a positive sign.

Example. What is the power factor in the circuit of the first problem on page 237 at the given frequency?

Solution. It is the cosine of $9° 9' = 0.987$.

If the inductance in henrys of a circuit is numerically equal to the capacity of the same in farads, there will be no reactance, and the tangent of ϕ will be expressed by $\frac{0}{R} = 0$, because the

reactance, which is the numerator of the expression for the tangent of ϕ, is of 0 value. The angle whose tangent is 0 is the angle of 0 degrees; hence when there is no reactance there is neither lag nor lead. The cosine of 0 degrees is 1.000; hence when there is no angle of lag or of lead the power factor is 1, and the power is given by the product of the effective e.m.f. by the effective current.

Angle of Lag and Rate of Change. The maximum rate of change of current value occurs when the value of cos θ is greatest, and this is when the sine of the curve is of zero value or when the sine function, in this case the current, is of zero value. By the law of inductance the product of the rate of change by the inductance of the circuit gives the counter e.m.f. This counter e.m.f. must be equal and opposed to the impressed e.m.f.; in other words the point of zero value of current must correspond with the point where the impressed e.m.f. has the value of the maximum counter e.m.f. of inductance and is opposite to it in polarity. If the current wave or cycle is taken as shifted backwards to bring about the equal opposed relation described, the angular distance through which it is shifted is called the angle of lag, and the maximum counter e.m.f. is evidently the sine of the angle of lag.

When two parts of an alternating circuit are in parallel with each other the combined impedance of the two branches is calculated in a way similar to that employed for the case of direct-current parallel circuits.

If the impedances on both branches are due to the one cause, i.e., to resistance, to capacity, or to inductance, the combined impedance is equal to the reciprocal of the sum of the reciprocals of the impedances exactly as in the case of parallel circuits carrying direct currents.

Example. Two branches of a circuit are in parallel, each being of negligible resistance and inductance. The capacity of one is

10 microfarads, the capacity of the other is 5 microfarads; the frequency is f. Calculate the combined reactance.

Solution. The reactances of the branches are respectively $\dfrac{1}{2\pi f \times 0.000,010}$ and $\dfrac{1}{2\pi f \times 0.000,005}$; the sum of their reciprocals is $2\pi f \times 0.000,015$, and the reciprocal of the sum of their reciprocals is $\dfrac{1}{2\pi f \times 0.000,015}$. This is simply the reactance of a capacity of the sum of the two capacities in parallel.

Placing capacities in parallel with each other is equivalent to the production of a capacity equal to the sum of the capacities in parallel with each other.

Example. Two inductances each of 0.03 henry are in parallel on a 125 frequency system. What is their combined reactance, assuming the resistances and capacities to be negligible in amount?

Solution. The reactance of a single inductance is $2\pi fL$, or $2\pi \times 125 \times 0.03 = 23.562$ ohms equivalent. Both are of the same value by the conditions of the problem; the sum of their reciprocals is $\dfrac{1}{23.562} + \dfrac{1}{23.562} = \dfrac{2}{23.562}$, the reciprocal of which is $\dfrac{23.562}{2} = 11.781$ ohmic equivalent.

The reactance of two equal inductances in parallel is one-half their sum.

Example. Let the two inductances be of 0.05 and 0.07 henry, the other conditions remaining the same as in the last problem. Calculate the combined reactance.

Solution. The reactance of the inductance 0.05 is $2\pi \times 125 \times 0.05 = 39.27$ ohmic equivalent; that of the inductance 0.07 is $2\pi \times 125 \times 0.07 = 54.978$. The sum of their reciprocals is $\dfrac{1}{39.27} + \dfrac{1}{54.978} = \dfrac{94.248}{2,158.986}$ and the reciprocal of this sum of the reciprocals is 22.9075 ohmic equivalent.

This problem is conveniently done by logarithms, thus:

log 39.270	1.594,061	39.270 + 44.978 = 94.248
log 54.978	1.740,189	
log of product	3.334,250	log of sum 94.248 as above 1.974,272

log of product	3.334,250
log of sum	1.974,272
log of $\frac{\text{product}}{\text{sum}}$	1.359,978

Ohmic equivalent = 22.9075.

Such problems can be done approximately with the slide rule.

For resistances in parallel in an alternating-current circuit the calculation is the same as for direct current.

Let an inductance and a resistance be in parallel with each other, and let an alternating e.m.f. be impressed upon them. It will be the same in all respects for both. Its instantaneous maximum value in terms of the inductance reactance will be $E_{max} = I'_{max}\, 2\,\pi f L$ (1), and in terms of the resistance of the other branch will be $E_{max} = I''_{max}\, R$ (2), I' indicating the current through the inductance and I'' the current through the resistance. The instantaneous values of the two currents are in quadrature with each other, so the resultant current is given by the expression $I_{max} = \sqrt{I'^2_{max} + I''^2_{max}}$ (3). From (1) and (2) we have $I'_{max} = \frac{E_{max}}{2\,\pi f L}$ and $I''_{max} = \frac{E_{max}}{R}$, and substituting in (3) these values gives

$$I_{max} = \sqrt{\frac{E^2_{max}}{(2\,\pi f L)^2} + \frac{E^2_{max}}{R^2}} = E_{max} \times \sqrt{\frac{1}{(2\,\pi f L)^2} + \frac{1}{R^2}} \quad (4)$$

From Ohm's law $I = E \times \frac{1}{R}$, so that it follows that $\sqrt{\frac{1}{(2\,\pi f L)^2} + \frac{1}{R^2}}$ is the reciprocal of the ohmic equivalent of the impedance of the parallel inductance and resistance,

and the ohmic equivalent of the impedance of the two branches is the reciprocal of the square root of the sum of the reciprocals of the squares of the resistance and reactance.

The same law applies to the reactance of capacity and is demonstrated in exactly the same way.

Example. Calculate the impedance of a resistance of 30 ohms in parallel with an inductance of 0.03 henry. The frequency is 100 cycles.

Solution. The inductance reactance is $2\pi \times 100 \times 0.03 = 18.85$ • Reciprocal of impedance =

$$\sqrt{\frac{1}{18.85^2} + \frac{1}{30^2}} = \sqrt{0.003,925} = 0.06265 \text{. Impedance} = \frac{1}{0.06265}$$

= 15.96 ohmic equivalent.

If capacity and resistance are in parallel, the impedance is equal to the reciprocal of the square root of the sum of the reciprocals of the squares of the resistance and capacity reactance.

Example. Calculate the impedance of a resistance of 20 ohms in parallel with a capacity of 90 microfarads with a frequency of 125.

Solution. The capacity reactance is $\frac{1}{2\pi \times 125 \times 0.000,090} =$ 14.147. The reciprocal of the impedance is

$$\sqrt{\left(\frac{1}{20}\right)^2 + \left(\frac{1}{14.147}\right)^2} = \sqrt{\frac{1}{400} + \frac{1}{200}} \text{ nearly} = 0.0816.$$

Therefore the impedance $= \frac{1}{0.0816} = 12.255$ the ohmic equivalent.

PROBLEMS.

What is the length of a sine curve ordinate at a point 0.8 radian from the origin or commencement of the base line? Take the value of a radian as 57.3°. *Ans.* 0.717.

If the base line is 3 inches long, where should the above ordinate be placed and how long should it be? In this case 2π is represented by

a length of 3 inches, and the radius of the generating circle in inch measurements is $3/2\pi$.

Ans. At 0.38 in. from origin; 0.34 in. high.

If the maximum e.m.f. of the above cycle is 2,250 volts, what is the voltage at the above point or period? *Ans.* 1,614 volts.

Calculate the average value of the e.m.f. of an alternating circuit of 1,275 volts maximum e.m.f. *Ans.* 811.665 volts.

If the average value of an alternating current is 97 amperes, what is its maximum value? *Ans.* 152.29 amperes.

What is the effective value of an alternating e.m.f. whose maximum value is 4,500 volts? *Ans.* 3,181.5 volts.

The effective value of an alternating current being 97.7 amperes, what is its maximum value? *Ans.* 138.15 amperes.

Calculate the reactance on a circuit due to an inductance of 0.172 henry and a frequency of 175. *Ans.* 189.12.

An alternating current requires 1/250 second for a cycle. The inductance of the circuit is 0.091 henry. Calculate the reactance of inductance. *Ans.* 142.94.

With an angular velocity due to 125 alternations per second and an inductance of 0.078 henry, apply the angular velocity formula to the calculation of the reactance of inductance in a circuit with the above frequency. *Ans.* ω = 785.4 radians; reactance = 61.26.

Calculate the rate of change of a 29-ampere (effective), 55 frequency current at 60°. *Ans.* 7,084 amperes.

Calculate the impedance due to a resistance of 179 ohms and an inductance of 0.11 henry with a frequency of 125. *Ans.* 199.

Calculate the impedance of a resistance of 251 ohms and an inductance of 0.901 henry with a frequency of 75. *Ans.* 493.

What is the angle of lag or of lead in the above case?

Ans. tan = 1.6916; angle of lag = 59° 24′.

With a resistance of 14 ohms, an inductance of 0.09 henry, a capacity of 31 microfarads, and a frequency of 75, calculate the impedance of a circuit and the angle of lag or of lead.

Ans. Impedance 29.56; tan = − 1.8601; angle of lead = 61° 44′.

In the last example let the maximum e.m.f. be 1,250 volts. Calculate the effective e.m.f. and current and the power factor and the power.

Ans. $E_{eff.}$ = 883.75 volts; $I_{eff.}$ = 29.89 amperes; power factor = 0.47358; power = 12,510 watts = 16.77 horse-power.

CHAPTER XVIII.

NETWORKS.

Cycles and Meshes. — Direction of Real Current. — Outer and Inner Meshes. — Indication of Cycles. — Cycle Equations. — Cycle Calculations. — Meaning of Cycle Letter. — Currents in a Network. — Maxwell's Rule for Writing the Equations of the Cycles. — Resistance of a Network. — Fleming's Method of Calculating the Resistance of a Network.

Cycles and Meshes. Any quantity can be expressed as the difference of two other quantities both of the same or of different signs. Thus 1 ampere can be expressed as the difference of 10 and 9 amperes or as 10–9 amperes. Instead of numbers letters can be used whose difference, as $x - y$, is taken as numerically equal to the real quantity. Thus the real quantity 1 ampere can be represented by $x - y$ if the condition is imposed on the unknown quantities that they shall differ by unity, or that $x - y$ shall equal 1.

Assume two meshes of electric conductors such as indicated in the cut, Fig. 1. Let them be connected at C and D to a source of e.m.f. Their relative resistances and connections may be such that a current will pass through AB or that none will pass. Assume in each mesh imaginary currents x and y circulating in the direction opposed to that of the hands of a watch and indicated by the curved arrows. Call this direction positive.

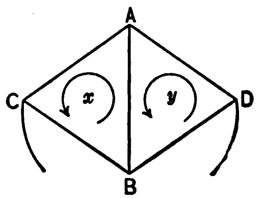


Fig. 1.

The intensity of the current through AB will be represented by the difference of the imaginary currents x and y. Then $x - y =$ the current in AB. Such imaginary currents are called "cycles."

If $x = y$, the real current in AB will be of zero intensity,

as in the balanced Wheatstone bridge. If $x > y$, there will be a current equal to $x - y$ from B to A. If $x < y$, there will be a current equal to $y - x$ from A to B.

The subtraction is to be made algebraically, giving each cycle its sign, positive or negative as the case may be. Thus in Fig. 2 let $x = 1$ and $y = -2$. Then $y - x = -2 - 1 = -3$, or the real current has a value of 3. On referring to Fig. 2, which represents this case, it is evident that the current in a lead such as AB lying between two cycles of opposite sign should have a value equal to their sum.

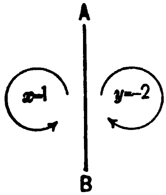


Fig. 2.

Direction of Real Current. The direction of the real current can be referred to the adjacent portions of one or both of the cycles. In Fig. 2 the current is from B to A, corresponding to that of both cycles. This is a case where the two cycles are of opposite sign. When both cycles have the same sign the direction of the real current corresponds to that of the numerically larger cycle. In Fig. 1, if $x = 1$ and $y = 2$, then the direction of the real current in AB corresponds to that of the adjacent side of y, or is from A to B.

The algebraic rule for direction follows from what has been said. Subtract algebraically one of the cycles from the other and refer the direction of the real current, whose value is thus found, to the minuend. If the real current has the same sign as the minuend its direction is the same as that of the adjacent part of the minuend. If of different sign from the minuend its direction is opposite.

Suppose we have two adjoining cycles $x = 5$ and $y = -5$. Taking x as the minuend we have

$$x - y = 5 + 5 = 10.$$

The value of 10 is positive, as is also the minuend; x gives the

direction of the real current. Taking y as the minuend we have

$$y - x = -5 - 5 = -10,$$

which tells us that the direction is the same as that of x. On looking at Fig. 2, which represents this condition, we see that the adjacent sides of both x and y coincide in direction, and both results are true and identical.

The meshes of Fig. 1 are shown without any source of e.m.f. Such source, a battery for instance, might be placed in any member. Or an outer source of e.m.f. may be introduced. If such is introduced it must inevitably introduce another mesh, as shown in Fig. 3. It is impossible to connect an outer source of e.m.f. to a mesh or meshes without thereby adding one more mesh to them.

In the diagram of the battery at E, Fig. 3, let the small bar represent the carbon or copper pole. Then the e.m.f. is of such polarity that it tends to produce a current from E to C, C to D, and D to E. The cycle x expresses this direction as drawn, and has a negative value because its direction is that of the movement of the hands of a watch. To the e.m.f. of a battery tending to

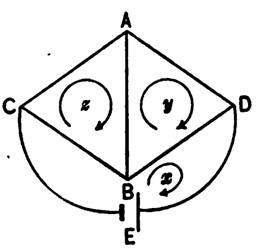


Fig. 3.

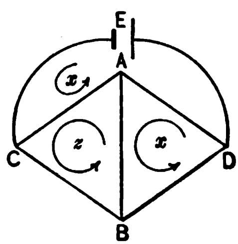


Fig. 4.

produce a negative cycle is to be assigned a negative sign. If this battery has an e.m.f. of 1.75 volts, it is written -1.75 for cycle calculations. The e.m.f. of a battery tending to produce a positive cycle has a positive sign.

If the battery mesh is supposed to be swung or rotated up-

ward as in Fig. 4 no change will occur in the relations of the real currents, but the cycles x and y become reversed and are now of positive sign. Therefore the battery becomes of positive e.m.f. and its voltage is written 1.75.

Outer and Inner Meshes. An outer mesh is one which has one or more sides on the outside border of the network. All the meshes in the diagrams Figs. 1, 3, and 4 are outer meshes. An inner mesh is one all of whose sides lie within the border of the network. In Fig. 5 an inner mesh is shown at A; the other four meshes are outer meshes. An inner lead is a conductor forming part of two meshes. The lead AB in Fig. 4 is an inner lead. An outer lead is one lying on the border of a network. CB and BD in Fig. 4 are outer leads.

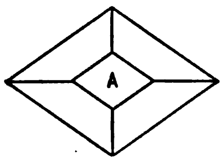


Fig. 5.

No cycle can be assumed to exist in space outside of a mesh. It follows that the current in the outer lead of a mesh is equal to the value of the cycle, because there is no cycle outside the network and adjacent to such lead, and therefore there is nothing to be added to or subtracted from the mesh cycle, which cycle therefore is the value of the real current.

Referring to Figs. 3 and 4 it is evident that the values and directions of the real currents are unaffected by the position of the active loop. In Fig. 3 the current in CA is equal to z, because CA is an outer lead, as explained in the last paragraph. In Fig. 4 the current in the same lead, CA, is equal to $x - z$, because CA is now an inner lead. It follows that the values of the cycles are different in the two diagrams, unless all the leads of $ACBD$ are of the same resistance. The value of network cycles depends upon the position of the active loop. This does not apply to the real current intensities, which are unaffected by such variations of position.

Indication of Cycles. Cycles are expressed by letters as symbols, like unknown quantities in algebra, as x, y, z, etc. They are used as the bases of simultaneous equations, one for each mesh, and therefore there are as many equations as there are meshes or cycles. One at least of the equations has a known quantity (e.m.f.) in the second term. Hence by the rules of algebra the equations can be solved, and the values of the unknown quantities, which are the imaginary currents in the cycles, are determined. By subtracting the cycle values from each other for inner leads, or by taking them unreduced for outer leads, the values of the real currents in the most intricate networks can be determined directly.

The signs of the cycles come out in the operation as $+$ or $-$ signs prefixed to the numerical values of the cycles as determined by means of the simultaneous equations. Suppose cycle x and cycle y lie next to each other. Their value is calculated. Assume that x proves to have a value of 5 and y to have a value of -5. Then we find the value of the current through the lead lying between x and y by simply adding the values of x and y and finding its direction by inspection of the diagram or algebraically by the signs.

The cycles x, y, etc., are all taken as of positive sign in the original equations. The known quantity or quantities to which the first member or members of one or more of the equations are equated are the voltages of the sources of e.m.f. acting on or taken as acting on the system and lying directly on one of the sides of a mesh, so that the mesh, if all other meshes were removed, would, with the generator, be an active electric circuit. This quantity is taken as positive if the polarity of the generator is such that it would tend to produce in its own mesh a cycle opposed in direction to the watch hands. If it would tend to produce a cycle of the same direction as the watch hands it is given a negative sign.

Cycle Equations. In writing a cycle equation the mesh inclosing the cycle is treated as a closed electric circuit. If a generator of e.m.f. is included in the circuit of the mesh, the first member of the equation of the cycle is equated to the value of the e.m.f. of that generator. If there is no generator in the circuit of the mesh, the first member of the equation of the cycle is equated to zero. Suppose that there are n meshes in a network and that there is a generator on an outer lead of an outer mesh. Then the cycle equation of that outer mesh will be equated to the e.m.f. of the generator and the remaining $n-1$ equations will be equated to zero. Suppose that the generator lies between two meshes of a network. Then the cycle equation of each of these meshes will be equated to the e.m.f. of the generator. For one equation the e.m.f. will be positive, for the other it will be negative, and there will be $n-2$ equations equated to zero. In all cases the sign to be prefixed to the e.m.f. of the generator is to be determined as by the considerations just given. An example of cycle calculations will illustrate what has been said above, and will show the significance of cycle values and of cycle signs.

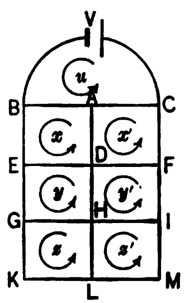


Fig. 6.

Cycle Calculations. The current in any lead will be indicated by its terminal letters large size, the resistance of any lead by its terminal letters small size. Thus in the lead AB, Fig. 6, the current would be indicated by the same letters, AB, the resistance by ab.

Assume the network indicated in the diagram Fig. 6. The battery at V is the source of e.m.f. As it tends to produce a cycle current of the opposite direction to that of the motion of the hands of a watch its e.m.f. has a positive sign given to it. Assume that every member of the network has the same resist-

ance, $ab = be = eg$, and so on. We know from Ohm's and Kirchhoff's laws the general distribution of current through the members of the network.

No current will go through AL because the circuit is balanced. This condition is expressed by stating that the left-hand cycles are equal respectively to the right-hand ones in pairs. Thus $x = x'$, $y = y'$, $z = z'$.

The lead BE is an outer lead; the real current in it is equal to x, or $BE = x$. The same considerations tell us that $EG = y$ and $GK = z$. We know from Kirchhoff's law that $BE > EG$ and that $EG > GK$. Therefore $x > y$ and $y > z$. The real current through ED is equal to $x - y$. As $x > y$ and as x is positive a current passes through ED from E to D, this direction being determined by the cycle of the minuend. This is in exact accord with Kirchhoff's laws, and the same process can be applied to other meshes, and results in accord with the known laws of the distribution of current will be obtained in every case.

The leads from the battery V to B and C are outer leads; therefore the current in them is equal to the cycle current u. By Kirchhoff's law u or $VB = BA + BE$. Applying cycles we have

$$u - x = AB,$$
$$x = BE.$$

Adding these we have $$u = AB + BE,$$

exactly in accordance with Kirchhoff's law.

In like manner the current x or BE divides itself between ED and EG by Kirchhoff's law, or

$$x \text{ or } BE = ED + EG.$$

Applying cycles we have

$$x - y = ED,$$
$$y = EG.$$

Adding these we have $$x = ED + EG,$$

as before, in accordance with Kirchhoff's law.

By applying this system of examination it would be found that by the application of cycles Kirchhoff's law is rigorously carried out.

Meaning of the Cycle Letter. The cycles, designated each by a single letter, are the basis of the determination of the currents in the different members of networks under given e.m.f. and of the determination of the total resistance of any network between given points. The letter designating the cycle means the number of amperes constituting the imaginary current of the cycle. Thus while it is convenient to speak of the cycle x, y, or a, as the case may be, the letters x, y, or a really stand each for a definite number of amperes. The value in amperes of each cycle is determined by simple algebraic methods next to be described. The knowledge of these values gives the distribution of currents in the network. If it is only required to know the resistance of a network, a definite e.m.f. is assumed to be connected to the points between which the resistance is required to be known. The generator and its connections make an extra mesh whose resistance is taken at zero. There is only one e.m.f., that of the generator, and the current through that mesh as a divisor divided into the e.m.f. as a dividend gives as quotient the resistance of the network.

Currents in a Network. To determine the currents in the leads of a network an equation has to be written for each mesh.

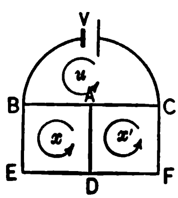


Fig. 7.

These are called cycle equations, or equations of the x cycle, of the y cycle, and so on. Assume the network represented in the diagram Fig. 7. Let the potentials at the points of intersection be designated by the letters at such points. Then $B-C$ would represent the fall in potential in the entire network below BC and including BC. By Ohm's law this is equal to the e.m.f. of the battery or gen-

erator at V less the product of the current in the battery leads VB and VC by the resistance of the battery and same leads. This resistance we will call v. It is the resistance from B to V to C, including leads and generator. The e.m.f. of the generator we will call e. The current in the leads VB and VC, which are outer leads, is equal to u. Then by Ohm's law, as stated above, we have

$$e - vu = B - C.$$

In this equation u indicates the current of the battery mesh.

The fall of potential in BA is equal to $B - A$. It is also equal to the product of the current in BA, which is $u - x$, by its resistance, which we have decided to term ba.

$$(u - x)(ba) = B - A.$$

By exactly similar process we find

$$(u - x')(ac) = A - C.$$

Transposing each equation we obtain

$$e = B - C + vu.$$
$$0 = A - B + (ba)(u - x).$$
$$0 = C - A + (ac)(u - x').$$

Adding these together we obtain

$$e = vu + ba(u - x) + ac(u - x').$$

Transposing and grouping this we find

$$u(v + ba + ac) - x(ba) - x'(ac) = e.$$

This is the equation of the cycle u.

The same result may be obtained in a shorter way. In the circuit $VBACV$ the currents are divisible into three; a current through CVB which is equal to u; one through BA which is equal to $u - x$; and one through AC which is equal to $u - x'$. By Ohm's law the sum of the e.m.f.'s in the circuit is equal to the product of resistances by currents, each

resistance being multiplied by the current passing through it. The e.m.f. in this circuit is e. Call the resistance of the generator and of its leads VB and VC simply v, and for the other resistances use the terminal letters of the respective leads, small size. We have therefore

$$vu + (ba)(u \mp x) + (ac)(u - x') = e.$$

Grouping this we have as before

$$u(v + ba + ac) - x(ba) - x'(ac) = e.$$

The equation for a cycle in a mesh which contains no source of e.m.f. in its circuit may next be deduced. Take the mesh $ABED$, whose cycle is designated as x. Assume a source of e.m.f. at any point, say at B, and let its value in volts be designated by e'. As the leads BE and ED are outer leads the current through them is equal to x. The current x multiplied by the resistances be and ed will, by Ohm's law, give the e.m.f. expended on BE and ED. If this is subtracted from the total e.m.f. of the circuit it will, by Kirchhoff's first law, give the e.m.f. expended on DAB, which, if we denote the potentials at the corners by letters placed there, is equal to $D-B$. This gives

$$e - x(be + ed) = D - B.$$

$x - x'$ is the current in AD. Multiplied by the resistance ad of the lead AD it gives the e.m.f. expended on that lead, and the parallel operation holds for AB giving

$$(x - x')\,ad = D - A,$$
$$(x - u)\,ab = A - B.$$

Transposing gives

$$(be + ed)\,x + D - B = e,$$
$$(ad)(x - x') + A - D = 0,$$
$$(ab)(x - u) + B - A = 0.$$

Adding and grouping we have

$$x(be + ed + ad + ab) - x'(ad) - u(ab) = e' = 0.$$

This is put equal to zero because there is no e.m.f. in the mesh.

This method may be applied as follows:

In the mesh $BEDA$, multiply the resistance of each lead by the current flowing through it, which gives, by Ohm's law, the e.m.f. expended on each of the leads. Added together they give the e.m.f. expended all around the circuit, which e.m.f. is zero, because there is no generator in the circuit. The equation thus obtained is

$$(be)\,x + (ed)\,x + (ad)\,(x - x') + (ab)\,(x - u) = 0.$$

Grouping this gives the same equation found above.

$$x\,(be + ed + ad + ab) - x'\,(ad) - u\,(ab) = 0.$$

We have now found cycle equations for a mesh with a generator in its circuit and for a mesh without one. Both are of identical form, and the rule for writing cycle equations is apparent. It is this:

Maxwell's Rule for Writing the Equations of the Cycles. Multiply the cyclic symbol by the sum of the resistances of the sides of its mesh; subtract from the result the products of the symbols of each of the adjoining cycles by the resistance common to it and to the mesh whose cyclic equation is to be written. Equate the result to the e.m.f. in the circuit of the cycle. Give to such e.m.f. a sign in accordance with the explanations given.

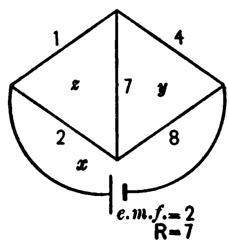


Fig. 8.

By applying this rule a cycle equation is written out for each mesh, and they are solved by regular alegbraic methods.

Example. Assume the network indicated in Fig. 8 in which the numbers indicate the resistances of the respective

leads. The numbers serve also to designate the leads. The resistance of the battery and its leads to A and B is 7 ohms. Calculate the current in the battery lead and in the leads 8 and 4.

Solution. Applying Maxwell's rule, we write out by inspection the three simultaneous equations of the cycles of the meshes x, y, and z.

$$\begin{aligned} x(7+8+2) - 8y \quad & - 2z &= 2. \quad (1) \\ -8x \quad + y(4+7+8) & - 7z &= 0. \quad (2) \\ -2x \quad - 7y \quad & + z(1+7+2) &= 0. \quad (3) \end{aligned}$$

In writing out these equations the same order is preserved for the three cycle symbols. Performing the indicated additions we have

$$17x - 8y - 2z = 2. \qquad (4)$$

$$-8x + 19y - 7z = 0. \qquad (5)$$

$$-2x - 7y + 10z = 0. \qquad (6)$$

Equations 1 and 4 are the equations of the cycle x; 2 and 5 the equations of the cycle y; and 3 and 6 the equations of the cycle z. The e.m.f. of the battery is positive because as placed the battery tends to produce a positive cycle as already explained.

Solving the above in the ordinary way or by determinants we find

$$x = 0.1936 \text{ ampere},$$

$$y = 0.129 \text{ ampere}.$$

As the battery leads are outer leads the current in them is equal to the cycle of their mesh, or to x. It is 0.1936 ampere.

The current in lead 4 being an outer lead is equal to the cycle of its mesh, or y. It is 0.129 ampere.

The current in lead 8 is equal to $x - y$, or $0.1936 - 0.129 = 0.0646$ ampere.

Example. Assume the network shown in the diagram Fig. 9. It comprises two sources of e.m.f. which are so connected that one of them tends to produce a negative cycle and the other one a positive cycle. Each has 2 volts e.m.f., and a positive sign is given to the battery E and a negative sign to the battery E'. Calculate the currents in the leads AEB and $AE'B$. The resistance of each battery with its leads is taken as 2 ohms, and the resistance of AB as 1 ohm.

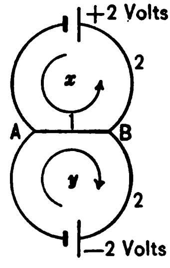


Fig. 9.

Solution. Writing out Maxwell's formulas by simple inspection of the diagram we have

$$x(2+1) - y = 2.$$

$$y(2+1) - x = -2.$$

Transposing and performing the indicated additions we have

$$3x - y = 2,$$

$$-x + 3y = -2,$$

which solved give

$$x = \frac{1}{2} \text{ and } y = -\frac{1}{2}.$$

As the battery leads are outer leads the current in each of them is equal to the cycle current in its mesh. This gives ½ ampere for each, and the direction is determined by the position of the battery in the circuit of the mesh. The current in the inner lead AB is equal to the difference of x and y.

$$x - y = \frac{1}{2} - \left(-\frac{1}{2}\right), \text{ or } \frac{1}{2} + \frac{1}{2} = 1.$$

Example. Assume the network of Fig. 10 which, after what has been said, is self-explanatory. Calculate the currents.

Solution. Writing out Maxwell's equations directly and adding and grouping them directly from the figure we have

$$x\,(2 + 2) - 2\,z = 2,$$
$$-\,2\,x - 2\,y + z\,(1 + 2 + 1 + 2) = 0,$$
$$y\,(2 + 2) - 2\,z = -\,2,$$

which solved give

$$x = \frac{1}{2},\ y = -\frac{1}{2},\ \text{and}\ z = 0.$$

$$\text{Current in } AB,\quad x - z = \frac{1}{2} - 0 = \frac{1}{2}\ \text{ampere.}$$

$$\text{Current in } A'B',\quad y - z = -\frac{1}{2} - 0 = -\frac{1}{2}\ \text{ampere.}$$

The values of x and y give the currents in the battery leads, because these leads are outer ones. The value of z, which is 0,

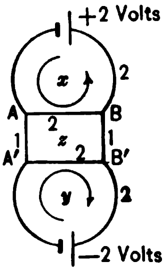


Fig. 10.

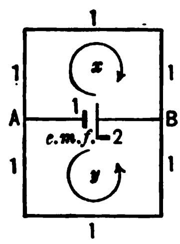


Fig. 11.

gives the currents in AA' and BB', because they are outer leads also. The currents in the battery leads are ½ ohm each, and the currents in AA' and BB' are of zero value.

Example. Assume the network of Fig. 11 and calculate the currents.

Solution. The e.m.f. to which the equation of the cycle x is equated has a negative sign; the e.m.f. for the y cycle has a positive sign. The equations are written out by simple inspection.

$$4x - \quad y = -2,$$
$$-x + 4y = 2,$$

which solved gives $x = -\frac{2}{5}$ and $y = \frac{2}{5}$, and the current through the intermediate lead AB is equal to $x - y = -\frac{2}{5} - \frac{2}{5} = -\frac{4}{5}$ of the sign of the minuend and corresponding therefore in direction with the portion of it adjacent to AB or from B to A. Or the subtraction may be made the other way, thus:

$$y - x = \frac{2}{5} - \left(-\frac{2}{5}\right) = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}.$$

The result has the sign of the minuend; its direction is therefore determined by y. On inspection it will be seen that both processes give it the same direction, which is as it should be, and its value is $\frac{4}{5}$ ampere.

The above equations may be solved by determinants. To do this a coefficient of 0 must be assigned to any cycle in an equation in which it normally would not appear, and this coefficient takes its place in the determinant. Thus the determinant of the three equations for Fig. 10 is the following:

For the common denominator,

$$\begin{vmatrix} 4 & 0 & -2 \\ -2 & -2 & 6 \\ 0 & 4 & -2 \end{vmatrix} = -64.$$

For the numerator of x,

$$\begin{vmatrix} 2 & 0 & -2 \\ 0 & -2 & 6 \\ -2 & 4 & -2 \end{vmatrix} = -32.$$

or $x = \frac{-32}{-64} = \frac{1}{2}$ as above, and so on for the others, y and z.

The reason for speaking so specifically of determinants is that they are the basis of a very simple rule for calculating directly and by a single operation the resistance of any network. The method is described in Fleming's "Handbook for the Electrical Laboratory and Testing Room," Vol. I, p. 204 et seq., and is given later in this chapter.

Resistance of a Network. The resistance of a network varies according to the points between which the resistance is taken. The resistance between any two points is thus determined: Assume that a generator of zero resistance is connected by leads of zero resistance to the points of the network between which the resistance is to be determined. Assigning any e.m.f. to the generator, the value of the current through the generator and leads is determined by cycle equations. But the only resistance through which this current has to go is that offered by the network. We therefore know the e.m.f. and the current; the resistance is calculated by Ohm's law.

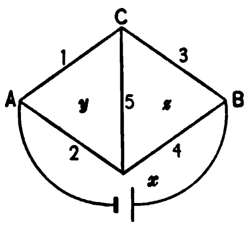


Fig. 12.

Example. What is the resistance between A and B of the network of Fig. 12, the resistance of each of whose members is placed alongside of its center?

Solution. Assume an e.m.f. say of 2 volts produced by a generator at V. The generator and its leads are assumed to have no resistance. They introduce a third mesh with its cycle x. Writing out by simple inspection the cycle equations we have

$$6x - 2y - 4z = 2.$$
$$-2x + 8y - 5z = 0.$$
$$-4x - 5y + 12z = 0.$$

Solving by ordinary algebraical methods so as to get the value of x we find

$$x = \frac{71}{85} \text{ ampere.}$$

As the generator leads are outer leads the current through them is of the value of the mesh cycle, or $\frac{71}{85}$ ampere. This is also the value of the current which goes through the network from A to B. As the battery and its leads are taken as of no resistance the resistance of the network between A and B is the total resistance of the circuit. Call this R. By Ohm's law we have

$$R = \frac{E}{I}.$$

In the problem to be solved $E = 2$ and $I = \frac{71}{85}$. Substituting these values in this equation we have

$$R = 2 \div \frac{71}{85} = 2 \times \frac{85}{71} = \frac{170}{71} = 2.394 \text{ ohms.}$$

The resistance of the network from A to B is 2.394 ohms.

Example. Calculate the resistance of the same network between D and C, Fig. 13.

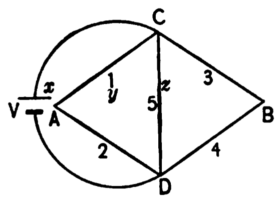


Fig. 13.

Solution. The e.m.f. may be taken at 1 volt. Writing out the cycle equations by inspection we have

$$\begin{aligned} 3x - 3y - 0z &= 1. \\ -3x + 8y - 5z &= 0. \\ -0x - 5y + 12z &= 0. \end{aligned}$$

Solving this gives

$$x = \frac{71}{105} \text{ ampere.}$$

By Ohm's law $R = \frac{E}{I}$. As $E = 1$, $R = 1 \div \frac{71}{105} = \frac{105}{71}$

$$= 1.479 \text{ ohms.}$$

The value of the e.m.f. of the assumed generator may be taken at any number of volts desired. Thus in the first example

it is taken at 2 and in the second at 1. The value 1 evidently simplifies the operation, as the reciprocal of the value of the cycle of the generator mesh becomes the value of the resistance of the network.

Example. Calculate the resistance of the same network between the points A and C.

Taking the e.m.f. as 1 volt write the equations as before.

$$\begin{aligned} x - y - 0z &= 1. \\ -x + 8y - 5z &= 0. \\ -0x - 5y + 12z &= 0. \end{aligned}$$

Solving for the value of x we find $x = \frac{71}{59}$ amperes and $R = 1 \div \frac{71}{59} = \frac{59}{71} = .8309$ ohm.

Fleming's Method of Calculating Networks. In each of the examples given to illustrate the use of Maxwell's cycles we have indicated by x the cycle of the mesh in whose circuit the generator lies. The second member of all the equations except one is zero, the second member of one equation; that of the mesh containing the generator is the e.m.f. of the generator. The rule for solving simultaneous equations of the first degree, or linear equations as they are often called, by the use of determinants is the following.

As a common denominator take the determinant of all the coefficients of the first members, in which members all the unknown quantities with their coefficients must be contained and in which no other quantity must be present. The unknown quantities must be written in the same order in all equations. For each unknown quantity a numerator is obtained by substituting for its coefficients in the other determinant the known quantities of the second member in order as written.

Any of the groups of equations we have just given is arranged in order, so that determinants can be written out from them.

Take the example on page 260. Writing out the coefficients of the first terms of the first members we have

$$\begin{vmatrix} 6 & -2 & -4 \\ -2 & 8 & -5 \\ -4 & -5 & 12 \end{vmatrix} = 496 - 326 = 170.$$

This is the determinant common denominator and is evaluated. As we wish to get the value of x or the generator cycle we substitute for the first or x column the terms 2, 0, and 0 of the second members of the three equations. This gives us the determinant for the numerator.

$$\begin{vmatrix} 2 & -2 & -4 \\ 0 & 8 & -5 \\ 0 & -5 & 12 \end{vmatrix} = (96 - 25) \times 2 = 71 \times 2 = 142.$$

Dividing the numerator of x, 142, by the denominator 170 we have

$$142 \div 170 = \frac{71}{85} \text{ ampere} = x.$$

Call the determinant first written which was the determinant of the denominator, Δn. Then the second determinant, as it has two zeros in its first column, is $2 \cdot \Delta(n - 1)$ and

$$x = \frac{2 \cdot \Delta(n - 1)}{\Delta n} = I.$$

But $R = E \div I$ and $E = 2$. Therefore

$$R = 2 \div \frac{2 \cdot \Delta(n - 1)}{\Delta n} = \frac{\Delta n}{\Delta(n - 1)}.$$

Whatever value may be assigned to the e.m.f. of the assumed generator it disappears in the equation which expresses the resistance, and we have a simple determinant expression for the resistance of any network. To obtain it proceed as follows:

Assume an extra mesh of z with a source of e.m.f. connected as just described to the network, which mesh and generator have no resistance. Call this the generator mesh.

Write out the equations of all the cycles of the network in order, putting all the x's in one column, all the y's in another, and so on. When a cycle letter does not enter into the equation, insert it with 0 for its coefficient. This has been done in the examples just given. The symbols of the generator mesh with their coefficients must be the first or left-hand members of the set of equations. This also has been carried out in the same examples. Write out the determinant of the coefficients and solve it and divide by the value of the first minor of its leading element. The quotient is the value of the cycle of the first terms; in all these examples this is the x cycle.

In each of such equations there will be one term which has for coefficient the sum of the resistances bounding one of the meshes. It will be a different mesh for each equation, and the unknown symbol in this term will indicate the mesh for whose cycle the equation is written. This gives a very simple way of writing out the determinants directly. It is thus given by Fleming:

Write as dexter diagonal of the determinant the sum of the resistances of each of the meshes of the network one by one, placing the resistance of the generator mesh in the upper left-hand corner. Complete the determinant by filling up each row with the resistances of the respective side or sides of meshes which separate this mesh from the one already written in its row. Give a minus sign to each of the latter. Where there is no intervening side or conductor place a zero in its place in the determinant. Apply the rule $\frac{\Delta n}{\Delta\ (n-1)}$ to this determinant, and the solution will give the resistance.

Example. Apply Fleming's rule to the network of Fig. 14. Each lead is supposed to have a resistance of 1; the bottom of the mesh y consists of 2 leads in series, giving the bottom a resistance of 2.

Solution. By the rule we must start with the assumed mesh x and write out the dexter diagonal of the determinant. The sum of the resistances of the bounding conductors of the mesh x is 2. That of y is 5, and z, y, u, and w have bounding conductors of resistance 3 each. It is well to write the column symbols in a row at the top before filling up the determinant. Starting with the diagonal and filling up as described we obtain

$$\begin{array}{c} \\ (1)\\ (2)\\ (3)\\ (4)\\ (5) \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & u & w \end{array} \\ \begin{vmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 5 & -1 & -1 & 0 \\ 0 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 3 \end{vmatrix} \end{array} = 98.$$

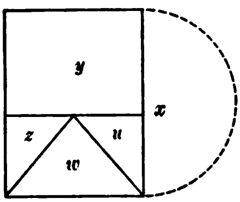


Fig. 14.

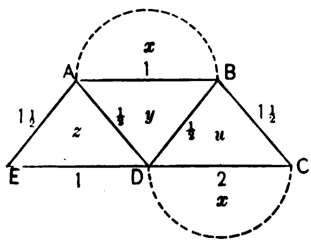


Fig. 15.

The minor of the leading element x of this determinant is

$$\begin{vmatrix} 5 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{vmatrix} = 87.$$

$$98 \div 87 = 1.126+ \text{ ohms}.$$

Example. Calculate by determinants the resistance of the network shown in Fig. 15 between the points A and B.

Solution. The resistances of the meshes x, y, z, and u are 1, 2, 3, and 4 ohms respectively. These give the diagonal row or dexter diagonal for the determinant, which is

$$\begin{array}{cccc} x & y & z & u \end{array}$$
$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 3 & 0 \\ 0 & -\frac{1}{2} & 0 & 4 \end{vmatrix} = 10\tfrac{1}{4}.$$

The first minor of the leading element of this determinant is

$$\begin{vmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 3 & 0 \\ -\frac{1}{2} & 0 & 4 \end{vmatrix} = 22\tfrac{1}{4}.$$

The resistance of the network between the points A and B is

$$10\tfrac{1}{4} \div 22\tfrac{1}{4} = 0.46 \text{ ohm}.$$

Example. Calculate the resistance of the same network between the points C and D.

Solution. The imaginary mesh x is now transferred and its resistance is that of the lead CD, or 2 ohms, and the determinant of the network is

$$\begin{array}{cccc} x & y & z & u \end{array}$$
$$\begin{vmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 3 & 0 \\ -2 & -\frac{1}{2} & 0 & 4 \end{vmatrix} = 21\tfrac{1}{2}.$$

The first minor of the leading element of this determinant is

$$\begin{vmatrix} 2 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 3 & 0 \\ -\frac{1}{2} & 0 & 4 \end{vmatrix} = 22\tfrac{1}{4}.$$

The resistance of the network between the points C and D is

$$21\tfrac{1}{2} \div 22\tfrac{1}{4} = 0.9663.$$

PROBLEMS.

Calculate the resistance of the network shown in Fig. 16 between the points A and B. Each side of a mesh is of resistance 1 ohm. *Ans.* $1\frac{1}{2}$ ohms.

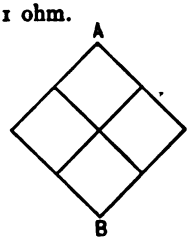


Fig. 16.

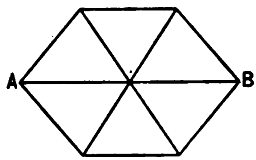


Fig. 17.

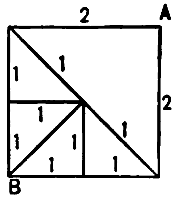


Fig. 18.

Calculate the resistance of the network of Fig. 17 between the points A and B. Each side of a mesh is of resistance 1 ohm.

Ans. $\frac{4}{5}$ ohm.

Calculate the resistance of the network of Fig. 18 between the points A and B. The resistance in ohms is marked upon the diagram. *Ans.* 1.833 or $1\frac{5}{6}$ ohms.

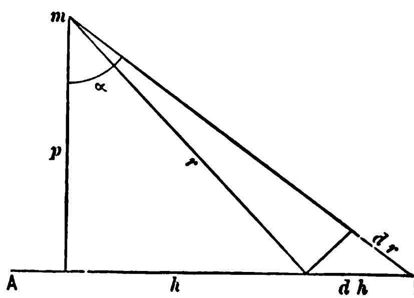
CHAPTER XIX.

DEMONSTRATIONS BY CALCULUS.

Force Exerted by Infinite Plane on a Point at Finite Distance.—Absolute Potential.—Average Value of Sine Functions.—Effective Value of Sine Functions.—Rate of Change.—Resistances of a Battery for Maximum Current.

Force Exerted by Infinite Plane on a Point at Finite Distance. Let AB represent the section of a plane and m represent a mass of attraction m. Let the attraction of the plane be represented by σ for a unit of its area. Then a unit area of the plane at unit distance will attract the mass with an attraction $m\sigma$. At the distance r the force of attraction between the two will be $f = \frac{m\sigma}{r^2}$ (1). The component of force attracting the mass to the plane is at any point the component perpendicular to the plane. Calling the angle between p and r, α, and calling the force perpendicular to the plane, f', we have, $f' = \frac{m\sigma}{r^2} \cos \alpha$ (2); and since $\cos \alpha = \frac{p}{r}$, $f' = \frac{m\sigma p}{r^3}$ (3).

The force f and the force f' will be constant over the annulus whose area is $2\,\pi\, hdh$. Reference to the diagram will explain this,



h being the radius of a circle drawn on the plane and $h + dh$ being the radius of a second circle concentric with the first, each having the end of p for a center. The annulus is included between these circles. From the diagram we see that $r^2 = h^2 + p^2$; p is a constant; therefore by differentiating we find $2\,hdh = 2\,rdr$, and $hdh = rdr$.

Substituting rdr for hdh in the expression $2\ \pi hdh$, we have as the area of the annulus $2\ \pi rdr$. The annulus is an infinitely small part of the area of the plane, and is therefore the differential of the area of the plane considered as a circle, or $ds = 2\ \pi rdr$ (4). The force exercised by this annulus at right angles to the plane is equal to the product of the area of the annulus by the force per unit area of the plane. Multiplying (3) by (4) gives $f'ds = 2\ \pi m\sigma p \frac{dr}{r^2}$ (5). Integrating this between the limits $r = \infty$ and $r = p$ gives: total force perpendicular to plane $= \int f'ds = 2\ \pi m\sigma p \int_{r=p}^{r=\infty} \frac{dr}{r^2} = 2\ \pi m\sigma p \left(\frac{1}{p} - \frac{1}{\infty}\right) = 2\ \pi m\sigma$.

Absolute Potential. Let e represent a charge of electricity concentrated at a point O. This is then a locus of attracting and repelling force. Let an indefinitely small body move from a point r cm. distant from O to a point r' cm. distant therefrom. Whatever path the body takes in doing this the net result will be a movement $r' - r$, taking r' as the greater distance.

The force acting on the body at the distance r is $\frac{e}{r^2}$. If moved a distance dr, the energy will be $\frac{e}{r^2}\ dr$. If moved a distance $r - r'$, the energy will be found by integrating this expression between the limits r' and r. This gives

$$\int_r^{r'} \frac{e}{r^2} dr = e\left(\frac{1}{r} - \frac{1}{r'}\right).$$

This is the energy involved in the movement of unit mass from r to r'. The absolute potential at the point r is found by substituting for r' the value infinity, which is equivalent to integrating between the limits $r = \infty$ and $r = r$. This gives as the value of the absolute potential of the point r, $e\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{e}{r}$.

Average Value of Sine Functions. In a sine curve let $x = E_{max}\omega t$, in which t is the time elapsed since the origin of the curve and in which ω is the rate of angular motion and E_{max} is the radius of the generating circle. As t and ω are both referred to time, their product, which is a product of time and velocity, gives a distance which is the abscissa of the curve. In the sine curve it is possible to refer the motion of a point along a straight line to an angular velocity, because the straight line is the development of a circle. This line is the axis of abscissas. Let E be the ordinate of the curve at any point. Then the area of a differential element of the area inclosed by half of the sine curve and axis of abscissas will be $Edx = E \times E_{max}\omega dt$. The rate of angular motion ω, it will be seen, is a constant, and t is a variable, so that $d\omega t = \omega dt$, and the differential equation is as above.

By the law of the sine curve $E = E_{max} \sin \omega t$, and substituting for E its value we have as the expression for the differential of the area A of the half-sine curve

$$dA = E^2_{max} \sin \omega t \times \omega\, dt, \qquad (1)$$

and integrating between the limits $\omega t = \pi$ and $\omega t = 0$ we have

$$A = E^2_{max} (- \cos \omega t)_0^\pi = - E^2_{max} (- 1 - 1) = 2\, E^2_{max},$$

because cos π and cos 0 are each numerically equal to 1.

As it is one-half of the area that is given by this expression, the area of the portion above or the portion below the line of abscissa, if it is divided by the length of the line forming the base of the curve, which line is the portion of the axis of abscissas in length $= \pi E_{max}$, or half the circumference of the generating circle, the quotient will be the average ordinate or the average value of the e.m.f. of an alternating system of the sine-curve type. This is

$$E \text{ (average)} = \frac{\text{area}}{\text{base}} = \frac{2\, E^2_{max}}{\pi E_{max}} = \frac{2\, E_{max}}{\pi} = 0.6366\, E_{max}.$$

The average value of a sine-type alternating current or e.m.f. is given by the above expression.

Effective Values of Sine Functions. If a curve is laid out with the base line of the sine curve as its base and with the squares of the ordinates of the sine curve as its ordinates, and if the area of such curve is divided by the base line, the quotient will be the average value of the ordinates, which is the average value of the squares of the ordinates of the sine curve. The square root of this quantity is the square root of the average value of the squares of the sines or ordinates of the sine curve or values of the alternating current. It is the effective value of the current.

The differential of the area of the new curve is given by the product of the square of an ordinate i^2 by the differential of the base line. The base line is equal to $I_{max}\ \omega dt$ in radian measurement, so for the differential we have

$$dA = i^2 \times I_{max}\ d\omega t, \tag{1}$$

and substituting for i its value in terms of I_{max}, which is I_{max} sin ωt, (1) becomes

$$dA = I^3_{max} \sin^2 \omega t d\omega t, \tag{2}$$

From trigonometry it is known that

$$\sin^2 \omega t = \frac{1 - \cos 2\ \omega t}{2},$$

and substituting this value in (2) and multiplying both numerator and denominator by 2 we find that we obtain a readily integrable expression:

$$dA = I^3_{max} \frac{2\ d\omega t - 2 \cos 2\ \omega t d\omega t}{4}, \tag{3}$$

whence by integrating between the limits $\omega t = \pi$ and $\omega t = 0$ we have

$$A = I^3_{max} \left[\frac{2\ \omega t - 2 \sin \omega t}{4}\right]_0^\pi. \tag{4}$$

The value of π is 180°, whose sine is 0; the sine of 0 is also 0, giving

$$A = I^2_{max}\frac{2\pi}{4} = \frac{\pi I^2_{max}}{2}. \tag{5}$$

The base of the area A is πI_{max}. Dividing (5) by this value gives the average value of the ordinates of the area,

$$\frac{A}{\pi I_{max}} = \frac{I^2_{max}}{2}, \tag{6}$$

and the square root of the second member of this equation is the square root of the average square of the ordinates of the sine curve, which is the effective or virtual current, or

$$I_{ef} = \frac{I_{max}}{\sqrt{2}} = 0.707\ I_{max}. \tag{7}$$

Rate of Change. The amount of change in the value of a sine current at any place is given by the difference between the values of two consecutive ordinates of the curve at that place multiplied by any coefficient required. The relative amount of change is then expressed by $(y + dy) - y = dy$. The time during which this change takes place is the difference of the corresponding abscissas, or $(x + dx) - x = dx$. As the differentials are those of current and of time they may be written di and dt. The rate of change is the quotient of change in value of current divided by the time required for such change to take place, or di/dt.

Let ω be the angular velocity of the current vector, which is the radians per second covered by the current alternations and in terms of frequency is equal to $2\pi f$, because a cycle is equal to 2π radians. The value of the current at any instant is given by the expression $I_{max} \sin \theta$. But θ is equal to ωt, θ being the degrees corresponding to t radians. Introducing this value of the

angle of position in the expression for the current value we have the equation

$$i = I_{\max} \sin \omega t,$$

in which i is the current value at any time ωt.

Differentiating this expression with respect to t we have

$$di = I_{\max} \cos \omega t d\omega t = \omega I_{\max} \cos \omega t dt,$$

and dividing by dt,

$$\frac{di}{dt} = \omega I_{\max} \cos \omega t = \text{rate of change at } t,$$

and substituting for ω its value, $2\pi f$, and for ωt its value in degrees, θ, we have

$$\text{Rate of change at } \theta = 2\pi f I_{\max} \cos \theta.$$

Resistance of a Battery for Maximum Current. Let n be the total number of cells in a given battery, and let r be the resistance of a single cell, and x be the number of cells in series. Then $\frac{n}{x}$ will be the number of cells in parallel, and by Ohm's law the resistance of the battery will be $rx \div \frac{n}{x} = \frac{rx^2}{n}$. Let R be the resistance of the outer circuit, and e be the e.m.f. of a single cell. Then the e.m.f. of the battery will be ex, and the current will be $ex \div \left(R + \frac{rx^2}{n}\right) = \frac{nex}{nR + rx^2}$. The value of R which will make this quantity a maximum is to be determined. The operation is simplified by using the reciprocal and finding the value of R which will make the reciprocal $\frac{nR + rx^2}{nex}$ a minimum. It is obvious that if an expression is a maximum, its reciprocal is a minimum. We then reduce the expression and omit the factor $\frac{1}{e}$ before differ-

entiating and then proceed by the regular process of maxima and minima,

$$\frac{nR + rx^2}{nex} = \frac{1}{e}\left(\frac{R}{x} + \frac{rx}{n}\right),$$

$$d\left(\frac{R}{x} + \frac{rx}{n}\right) = \frac{nrdx}{n^2} - \frac{Rdx}{x^2},$$

and taking the differential coefficient equal to zero, we find the value of R which makes the reciprocal of the current a maximum or a minimum; thus $\frac{r}{n} - \frac{R}{x^2} = 0$ and $R = \frac{rx^2}{n}$.

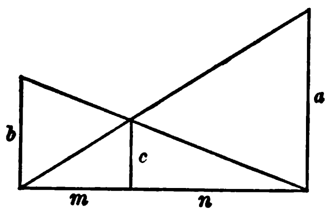
To determine whether this value of R gives a maximum or minimum value of current we must find the sign of the second differential coefficient. Proceeding by the regular methods we find $d\left(\frac{r}{n} - \frac{R}{x^2}\right) = 2\ Rxdx$, giving the second differential coefficient, $2\ Rx$, which is positive. Therefore the value $R = \frac{rx^2}{n}$ gives a minimum value to the reciprocal of the current or a maximum value of current. But we have seen that $\frac{rx^2}{n}$ is the internal resistance of the battery. Therefore the arrangement of cells which will make the external and internal resistance equal will give the maximum current.

APPENDIX A.

GEOMETRICAL SOLUTION OF PARALLEL CIRCUITS.

The resistance of two conductors in parallel can be determined by a geometrical construction.

On the extremities of any line as a base erect two perpendicular lines, a and b, proportional to the resistance of the two conductors. Draw diagonals as shown from the upper end of each line to the base of the other.



From the point of intersection of the diagonals drop a perpendicular c to the base line. The length of c is proportional to the combined resistance of the two conductors.

For this to be true, the value of c must be given by the following equation:

$$c = \frac{a \times b}{a + b} \text{ (see page 65).}$$

This is proved as follows:

Let c divide the base line into the two parts m and n. Then

$$\frac{a}{c} = \frac{m+n}{n}. \qquad (1)$$

$$\frac{b}{c} = \frac{m+n}{m}. \qquad (2)$$

$$\frac{a}{b} = \frac{m}{n}. \qquad (3)$$

Dividing (1) by (2) $\quad \frac{a+b}{b} = \frac{m+n}{n}. \qquad (4)$

by composition substituting from (1)

$$\frac{a+b}{b} = \frac{a}{c}. \qquad (5)$$

which gives $\quad c = \frac{ab}{a+b}. \qquad (6)$

Q.E.D.

If the resistance of three or more conductors in parallel is required the same method can be applied.

Erect on the horizontal line a perpendicular line for each of the resistances, the lengths of the lines being proportional to the resistances which they severally represent. Combine any two of them by the method just described. Then, following exactly the same method, combine the resistance represented by the new perpendicular with that of one of the remaining perpendiculars.

This gives the combined resistance of three conductors. The process is followed out until the resistances have all been used. The final result will be a perpendicular line much shorter than any of the constituents and whose length will be proportional to the combined resistance of all the resistances represented by the perpendiculars originally drawn.

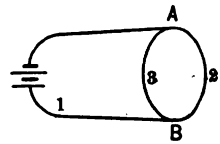
As is the case with many other graphic methods, the accuracy of this process is not very great, and it naturally becomes less accurate as more resistances have to be combined. This is because the line proportional to and representing the combined resistances soon becomes too short for accurate measurement.

APPENDIX B.

ALGEBRAIC SOLUTION OF CIRCUITS.

Elementary Cases of Net Works. Some typical circuits are treated algebraically in the following pages. By the application of Kirchhoff's and Ohm's laws and the use of algebra, the formulas for their solutions are deduced. They can also be solved by Maxwell's cycles, as explained in Chapter XVIII.

Example. Calculate the distribution of currents in the circuit shown in the diagram.



Solution. The circuit is treated as of three branches, 1, 2 and 3. Branch 1 includes the generator. Subscript letters refer to the respective branches, r indicating resistance and i indicating current. The combined resistance of 2 and 3 is $\frac{r_2 r_3}{r_2 + r_3}$ (see page 65) (1).

The total resistance of the circuit is

$$r_1 + \frac{r_2 r_3}{r_2 + r_3} = \frac{r_1 r_2 + r_1 r_3 + r_2 r_3}{r_2 + r_3}. \qquad (2)$$

Let E be the e.m.f. of the generator. By Ohm's law $(I = E/R)$

$$i = E \div \frac{r_1 r_2 + r_2 r_3 + r_1 r_3}{r_2 + r_3} = \frac{r_2 + r_3}{r_1 r_2 + r_2 r_3 + r_1 r_3} \times E. \qquad (3)$$

The e.m.f. expended in each of the two branches, 2 and 3, is identical in amount by Kirchhoff's law. It is equal to the combined resistance of the two branches divided by the total resistance, the quotient being multiplied by the total e.m.f. of the system. This gives

$$e_2 \text{ or } e_3 = \frac{r_2 ; r_3}{R} \times E = \frac{r_2 r_3}{r_2 + r_3} \div \frac{r_1 r_2 + r_2 r_3 + r_1 r_3}{r_2 + r_3} \times E$$

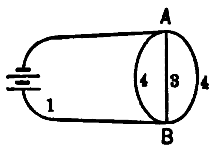
$$= \frac{r_2 r_3}{r_1 r_2 + r_2 r_3 + r_1 r_3} \times E. \qquad (4)$$

By Ohm's law $i_2 = \frac{e_2}{r_2}$ (5), and $i_3 = \frac{e_3}{r_3}$ (6).

Substituting in (5) and (6) the values of e_2 and e_3 from (4) gives

$$i_2 = \frac{r_3}{r_1r_2 + r_2r_3 + r_1r_3} \times E \ (7) \text{ and } i_3 = \frac{r_2}{r_1r_2 + r_2r_3 + r_1r_3}. \quad (8)$$

These expressions for the resistances of the branches 2 and 3 have the same denominators, therefore they have the ratio of their numerators, which ratio is the inverse ratio of the resistances of the branches, or $i_2 : i_3 :: r_3 : r_2$.



Example. Assume three branches, 2, 3 and 4, starting from A, and coming together at B. Calculate the currents.

Solution. Proceeding as before, we have $R = r_1 + (r_2; r_3; r_4) =$

$$r_1 + \frac{r_2r_3r_4}{r_2r_3 + r_2r_4 + r_3r_4} = \frac{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}{r_2r_3 + r_2r_4 + r_3r_4}. \quad (9)$$

This is the total resistance of the circuit. The distribution of current has now to be determined. Proceeding as before,

$$e_2, e_3, \text{ or } e_4, = E \times \frac{r_2 : r_3 ; r_4}{R} = \frac{E}{R} \times \frac{r_2r_3r_4}{r_2r_3 + r_2r_4 + r_3r_4}, \quad (10)$$

and substituting for R in this equation its value from (9), and reducing by carrying out the indicated operations, we obtain

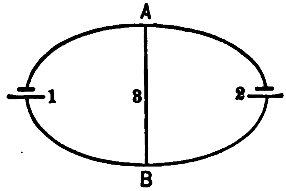
$$e_2, e_3, \text{ or } e_4, = E \times \frac{r_2r_3r_4}{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}. \quad (11)$$

As $i_2 = \frac{e_2}{r_2}$ (12), $i_3 = \frac{e_3}{r_3}$ (13), and $i_4 = \frac{e_4}{r_4}$ (14), (11) divided by r_2, r_3 or r_4 will give the respective currents in branches 2, 3 and 4. Thus for branch 2, divide (11) by r_2, giving:

$$\frac{e_2}{r_2} = i_2 = E \times \frac{r_3r_4}{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}. \quad (15)$$

For branches 3 and 4 the values of the currents would differ only in the numerators; for branch 3, the numerator would be, r_2r_4, and for branch 4, the numerator would be, r_2r_3.

Example. Calculate the distribution of current in the circuit shown in the diagram.



Solution. The branches 1 and 2 include generators. The symbols correspond with those of the preceding example.

The e.m.f. of the generator of branch 2, expended in branches 1 and 3, taking account only of the resistance, and omitting consideration of the counter e.m.f., is

$$e_1 \text{ or } e_3 \text{ (of generator 2)} = E_2 \times r_1\,;\, r_3 \div (r_2 + r_1\,;\, r_3) =$$

$$E_2 \times \frac{r_1r_3}{r_1 + r_3} \div \frac{r_1r_2 + r_1r_3 + r_2r_3}{r_1 + r_3} = E_2 \times \frac{r_1r_3}{r_1r_2 + r_1r_3 + r_2r_3}. \quad (1)$$

Going through the same operation for the e.m.f. in branches 2 and 3, due to generator 1, we obtain

e_2 or e_3 (of generator 1) $= E_1 \times r_2;\, r_3 \div (r_1 + r_2;\, r_3)$ giving, as above,

$$E_1 \times \frac{r_2r_3}{r_1r_2 + r_1r_3 + r_2r_3}. \quad (2)$$

If the two generators are arranged as indicated in the diagram, they will work together as far as the branch 3 is concerned, and the e.m.f. of 3 will be the sum of the second members of (1) and (2), or

$$e_3 = \frac{E_1r_2r_3 + E_2r_1r_3}{r_1r_2 + r_1r_3 + r_2r_3}. \quad (3)$$

Applying Ohm's law, $i_3 = \frac{e_3}{r_3}$, divide the above equation by the resistance of 3, which is r_3. This gives

$$i_3 = \frac{e_3}{r_3} = \frac{E_1r_2 + E_2r_1}{r_1r_2 + r_1r_3 + r_2r_3}. \quad (4)$$

If the generators are in opposition as regards branch 3, the current in 3 will be

$$i_3 = \frac{E_1 r_2 - E_2 r_1}{r_1 r_2 + r_1 r_3 + r_2 r_3}. \tag{4a}$$

The e.m.f. of branch 2, due to its own generator, is

$$e_2 \text{ (produced by generator 2)} = E_2 \times r_2 \div (r_2 + r_1; r_3)$$

$$= E_2 \times \frac{r_2 (r_1 + r_3)}{r_1 r_2 + r_1 r_3 + r_2 r_3}. \tag{5}$$

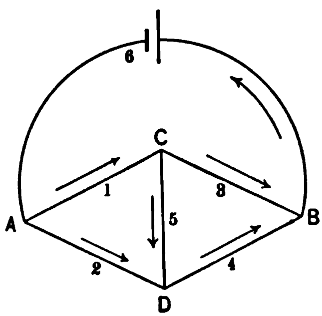
The e.m.f. of branch 2, due to the generator 1, is given by equation (2). Subtracting the last member of (2) from the last member of (5) gives the actual e.m.f. of the branch, 5, because one generator as shown is working in opposition to the other. Then, applying Ohm's law,

$$i_2 = \frac{e_2}{r_2} = \frac{E_2 (r_1 + r_3) - E_1 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3}. \tag{6}$$

Exactly similar processes give the value of the current in 1:

$$i_1 = \frac{e_1}{r_1} = \frac{E_1 (r_2 + r_3) - E_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3}. \tag{7}$$

Had one of the batteries or generators been connected with opposite polarity to that shown, equations (6) and (7), with a plus sign in place of the minus sign between the two terms of the numerator, would give the value of the currents.



Example. Calculate the currents in the "Wheatstone bridge" circuit, shown in the diagram.

Solution. Let the arrows denote the direction of the currents. By Kirchhoff's first law

$$i_3 = i_1 - i_5, \quad (1); \quad i_4 = i_2 + i_5, \quad (2); \quad i_6 = i_1 + i_2, \quad (3)$$

By Ohm's law the e.m.f. of branches 3 and 6 is obtained by multiplying both members of (1) and (3) by r_3 and r_6 respectively.

$$r_3 i_3 = r_3 i_1 - r_3 i_5 \quad (4); \qquad r_6 i_6 = r_6 i_1 + r_6 i_2. \tag{5}$$

The entire e.m.f. of the system is expended on the circuit 1, 3 and 6. Adding the second members of (4) and (5) to $r_1 i_1$, the latter, the e.m.f. of 1, gives the total e.m.f. of the system, because such sum is the sum of the e.m.f.'s of the three branches, 1, 3 and 6. Thus:

$$E = r_3 i_1 - r_3 i_5 + r_6 i_1 + r_6 i_2 + r_1 i_1 = i_1 (r_3 + r_6 + r_1) + r_6 i_2 - r_3 i_5. \tag{6}$$

The e.m.f. expended in the branch 2 is equal to the sum of the e.m.f.'s of 1 and of 5, giving

$$r_2 i_2 = r_1 i_1 + r_5 i_5 \text{ or } r_1 i_1 - r_2 i_2 + r_5 i_5 = 0. \tag{7}$$

The e.m.f. of 3 is given by adding that of 4 to that of 5, or

$$r_3 i_3 = r_4 i_4 + r_5 i_5. \tag{8}$$

Substituting for $r_3 i_3$ in this equation, its value from (4), gives:

$$r_3 i_1 - r_3 i_5 = r_4 i_4 + r_5 i_5. \tag{9}$$

The current in 4 is equal to the sum of the currents in 5 and 2, giving as in the other parallel processes

$$i_4 = i_2 + i_5, \tag{10}$$

and

$$r_4 i_4 = r_4 i_2 + r_4 i_5. \tag{11}$$

Substituting this value of $r_4 i_4$ in (9) gives

$$r_3 i_1 - r_3 i_5 = r_4 i_2 + r_4 i_5 + r_5 i_5, \tag{12}$$

and transposing and factoring,

$$r_3 i_1 - r_4 i_2 - i_5 (r_3 + r_4 + r_5) = 0. \tag{13}$$

In (6), (7) and (13) three currents appear, i_1, i_2 and i_5. We next proceed to eliminate i_1 and i_2 from them, treating them as simultaneous equations.

Multiply (7) by r_3. This gives

$$i_1 r_1 r_3 - i_2 r_2 r_3 + i_5 r_3 r_5 = 0. \tag{14}$$

Multiply (13) by r_1. This gives

$$i_1 r_1 r_3 - i_2 r_1 r_4 - i_5 (r_1 r_3 + r_1 r_4 + r_1 r_5) = 0. \qquad (15)$$

Subtracting (14) from (15) gives

$$i_2 (r_2 r_3 - r_1 r_4) - i_5 (r_1 r_3 + r_1 r_4 + r_1 r_5 + r_3 r_5) = 0. \qquad (16)$$

Dividing by $(r_2 r_3 - r_1 r_4)$ and transposing, gives

$$i_2 = i_5 \frac{r_1 r_3 + r_1 r_4 + r_1 r_5 + r_3 r_5}{r_2 r_3 - r_1 r_4}. \qquad (17)$$

Substitute this value of i_2 in (7) and transpose.

$$r_1 i_1 = i_5 \frac{r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_2 r_5 + r_2 r_3 r_5}{r_2 r_3 - r_1 r_4} - r_5 i_5. \qquad (18)$$

Divide both members by r_1, and bring $- r_5 i_5$ into the numerator by multiplying it by $r_2 r_3 - r_1 r_4$. This gives

$$i_1 = i_5 \frac{r_2 r_3 + r_2 r_4 + r_2 r_5 + \dfrac{r_2 r_3 r_5}{r_1} - \dfrac{r_2 r_3 r_5}{r_1} + r_4 r_5}{r_2 r_3 - r_1 r_4}. \qquad (19)$$

The fourth and fifth terms of the numerator cancel each other. Substituting the values of i_2 and i_1 from (17) and (19) in (6), we obtain

$$E = i_5 \frac{r_2 r_3 + r_2 r_4 + r_2 r_5 + r_4 r_5}{r_2 r_3 - r_1 r_4} \times (r_1 + r_3 + r_6)$$
$$+ i_5 \frac{r_1 r_3 + r_1 r_4 + r_1 r_5 + r_3 r_5}{r_2 r_3 - r_1 r_4} r_6 - r_3 i_5. \qquad (20)$$

Bring $r_3 i_5$ into the numerator by multiplying it by $r_2 r_3 - r_1 r_4$, perform the indicated multiplications, and arrange the terms of the numerator in numerical order. The numerator then becomes

$r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_2 r_5 + r_1 r_4 r_5 + r_2 r_3 r_3 + r_2 r_3 r_4 + r_2 r_3 r_5 + r_3 r_4 r_5 + r_2 r_3 r_6 + r_2 r_4 r_6 + r_2 r_5 r_6 + r_4 r_5 r_6 + r_1 r_3 r_6 + r_1 r_4 r_6 + r_1 r_5 r_6 + r_3 r_5 r_6 - r_2 r_3 r_3 + r_1 r_3 r_4.$

In this expression the fifth and seventeenth terms cancel each other, and by factoring the expression becomes

$r_5 r_6 (r_1 + r_2 + r_3 + r_4) + r_5 (r_1 + r_3) (r_2 + r_4) + r_6 (r_1 + r_2) (r_3 + r_4) + r_1 r_3 (r_2 + r_4) + r_2 r_4 (r_1 + r_3),$

which expression we will call D.

Then write out (20) as modified, using the symbol D to express the numerator, and we have

$$E = i_5 \frac{D}{r_2 r_3 - r_1 r_4}, \tag{21}$$

dividing by $\frac{D}{r_2 r_3 - r_1 r_4}$, and transposing,

$$i_5 = \frac{E\,(r_2 r_3 - r_1 r_4)}{D}. \tag{22}$$

This is the current in branch 5.

Transposing (3) gives

$$i_2 = i_6 - i_1. \tag{23}$$

Substituting this value of i_2 in (7) and factoring, gives

$$r_1 i_1 - r_2 i_6 + r_2 i_1 + r_5 i_5 = i_1 (r_1 + r_2) + r_5 i_5 - r_2 i_6 = 0. \tag{24}$$

Doing the same in (13), gives

$$r_3 i_1 - (i_6 - i_1)\, r_4 - i_5 (r_3 + r_4 + r_5) = i_1 (r_3 + r_4)$$
$$- i_5 (r_3 + r_4 + r_5) - r_4 i_6 = 0. \tag{25}$$

Eliminate i_1 from (24) and (25), by first transposing and dividing (24) by $r_1 + r_2$, and (25) by $r_3 + r_4$ respectively, which gives

$$i_1 = \frac{r_2 i_6 - r_5 i_5}{r_1 + r_2} \text{ (26), and } i_1 = \frac{r_4 i_6 + i_5 (r_3 + r_4 + r_5)}{r_3 + r_4}, \tag{27}$$

and then complete the elimination by equating the second members:

$$\frac{r_2 i_6 - r_5 i_5}{r_1 + r_2} = \frac{r_4 i_6 + i_5 (r_3 + r_4 + r_5)}{r_3 + r_4}. \tag{28}$$

Clear of fractions, perform the indicated multiplications, and factor. The term $r_2 r_4 i_6$ cancels out in the operations, and transposing we obtain

$$i_6 (r_2 r_3 - r_1 r_4) = r_5 i_5 (r_1 + r_2 + r_3 + r_4) + i_5 (r_1 + r_2)(r_3 + r_4); \tag{29}$$

whence we obtain

$$i_6 = \frac{i_5 [r_5 (r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)]}{r_2 r_3 - r_1 r_4}. \tag{30}$$

Substituting for i_5 its value from (22), the binomial $r_2r_3 - r_1r_4$ disappears, and we obtain

$$i_6 = \frac{E\,[r_5\,(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)\,(r_3 + r_4)]}{D}. \qquad (31)$$

This is the current in branch 6.

Substitute in (26) the values of i_5 from (22), and of i_6 from (31).

$$i_1 = \frac{E\,[r_5\,(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)\,(r_3 + r_4)]\,r_2 - E\,r_5\,(r_2r_3 - r_1r_4)}{D\,(r_1 + r_2)}. \qquad (32)$$

Performing the indicated multiplications, factoring, and reducing, with the disappearance of the terms $r_2r_3r_5$ and $r_1 + r_2$ we obtain

$$i_1 = \frac{E\,(r_2r_3 + r_2r_4 + r_2r_5 + r_4r_5)}{D}. \qquad (33)$$

This is the current in the branch 1.

From equation (3) we find, by transposing,

$$i_2 = i_6 - i_1, \qquad (34)$$

and substituting for i_6 and i_1 their values from (31) and (33), performing all indicated operations, simplified by the fact that both fractions have the same denominator, we obtain

$$i_2 = \frac{E\,(r_1\,r_5 + r_3r_5 + r_1r_3 + r_1r_4)}{D}. \qquad (35)$$

This is the current in branch 2.

Equation (1) reads $i_3 = i_1 - i_5$. Substituting in it for i_1 and i_5 their values from (22) and (33), and reducing as hitherto, we find

$$i_3 = \frac{E\,(r_2r_4 + r_2r_5 + r_4r_5 + r_1r_4)}{D}. \qquad (36)$$

This is the current in branch 3.

From equation (2), $i_4 = i_2 + i_5$; substituting the values of i_2 and i_5 from (22) and (35), and reducing, we find

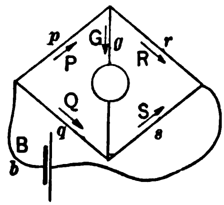
$$i_4 = \frac{E\,(r_1r_5 + r_3r_5 + r_1r_3 + r_2r_3)}{D}. \qquad (37)$$

This is the current in branch 4.

APPENDIX C.

WHEATSTONE BRIDGE LAW.

The Wheatstone bridge law may be thus deduced by the use of determinants. The usual diagram represents the Wheatstone bridge connections, and in it capital letters indicate the currents flowing through the arms of the bridge and through the galvanometer connection, and the small letters indicate resistances of the same parts.



From Kirchhoff's first law we have the three relations

$$B = P + Q, \tag{1}$$

$$R = P - G, \tag{2}$$

$$S = Q + G. \tag{3}$$

These equations refer to the currents.

The e.m.f. at different points of the bridge is given by Kirchhoff's second law, giving the three equations

$$Gg + Pp - Qq = 0, \tag{4}$$

$$-Gg + Rr - Ss = 0, \tag{5}$$

$$Bb + Qq + Ss = E. \tag{6}$$

in which last equation E represents the e.m.f. of the battery. Substituting the values of B, R, and S in equations (4), (5), and (6) from equations (1), (2), and (3) we have

$$Gg + Pp - Qq = 0, \tag{7}$$

$$-Gg + (P - G)r - (Q + G)s = 0, \tag{8}$$

$$(P + Q)b + Qq + (Q + G)s = E. \tag{9}$$

These we may rewrite as

$$Gg + Pp - Qq = 0, \tag{10}$$

$$-G(g + r + s) + Pr - Qs = 0, \tag{11}$$

$$Gs + Pb + Q(b + q + s) = E. \tag{12}$$

Solving this by the regular determinant method we have for the general divisor determinant

$$\begin{vmatrix} g, & p & -q \\ -(g + r + s) & r & -s \\ s & b & b + q + s, \end{vmatrix}$$

whose value we may indicate by D. The numerator determinant for the value of G:

$$\begin{vmatrix} 0 & p & -q \\ 0 & r & -s \\ 1 & b & (b + q + s). \end{vmatrix}$$

The evaluation of this determinant is $qr - ps$, giving us for the value of the galvanometer current, G,

$$G = \frac{E(qr - ps)}{D}. \tag{13}$$

From equation (13) it follows for G to be equal to zero that $qr - ps$ must be equal to zero, or $qr = ps$, which put in the form of a proportion gives the proportional form of the law of the Wheatstone bridge,

$$q : p :: s : r.$$

TABLES OF EQUIVALENTS

LENGTH.

1 in.	=	25.40010 mm.
1 ft.	=	0.30480 Meter
1 yd.	=	0.91440 Meter
1 mile	=	1.60935 Km.
1 Nautical Mile	=	1853.25 Meters
1 fathom	=	1.829 Meters

1 Meter	=	39.37043 In.
1 Meter	=	3.28083 Ft.
1 Meter	=	1.09361 Yds.
1 Km.	=	0.62137 Mile

AREA.

1 Sq. In.	=	6.452 Sq. cm.
1 Sq. Ft.	=	9.290 Sq. dm.
1 Sq. Yd.	=	0.836 Sq. M.
1 Sq. Mile	=	259.008 Hectares

1 Sq. cm.	=	0.1550 Sq. In.
1 Sq. M.	=	10.764 Sq. Ft.
1 Sq. M.	=	1.196 Sq. Yd.

WEIGHT.

1 grain	=	64.7989 mg.
1 oz. Av.	=	28.3495 Gm.
1 oz. Troy	=	31.10348 Gm.
1 lb. Av.	=	0.45359 Kilo.
1 lb. Troy	=	0.37324 Kilo.
1 lb. Av.	=	453.5924 Gm.

1 mg.	=	0.01543 grain
1 Gm.	=	15.43236 grains
1 Kilo.	=	33.814 flu. oz.
1 Kilo.	=	2.20462 lb. Av.
1 Kilo.	=	2.67924 lb. Troy
1 Kilo.	=	35.274 oz. Av.
1 Kilo.	=	32.1507 oz. Troy
1 Millier or Tonne	=	2204.62 lb. Av.
1 Quintal	=	220.462 lb. Av.

VOLUME.

1 minim (water)	= 0.06161	c.c.
1 flu. dr.	= 3.70	c.c.
1 flu. oz.	= 29.5737	c.c.
1 Apoth. oz. (water)	= 31.10348	c.c.
1 quart	= 0.94636	Liter
1 U. S. gal.	= 3.78543	Liters
1 bushel	= 0.35239	Hectol

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1 c.c.	= 16.23	minims (water)
1 c.c.	= 0.2702	flu. dr.
1 Centiliter	= 0.338	flu. oz.
1 Liter	= 1.0567	qt.
1 Liter	= 0.26417	gal.
1 Decaliter	= 2.6417	gal.
1 Hectoliter	= 2.8377	bushels

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1 cu. in.	= 16.387	c.c.
1 cu. ft.	= 0.02832	c. M.
1 cu. yd.	= 0.765	c. M.

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1 c.c.	= 0.06102	cu. in.
1 c. dm.	= 61.023	cu. in.
1 c. M.	= 35.314	cu. ft.
1 c. M.	= 1.308	cu. yd.

FORCE.

1 Poundal	= 13,825 dynes
1 Pound	= 4.45×10^5 dynes
1 Grain	= 63.6 dynes
1 Gram	= 981 dynes

ENERGY.

1 foot-pound	= 13,823 gram-centimeters	1.3560×10^7 ergs
1 foot-poundal	=	4.214×10^5 ergs
1 foot-ton	= 3.096×10^7 gram-centimeters	3.0374×10^{10} ergs
1 joule	=	10^7 ergs

POWER, ENERGY RATE, OR ACTIVITY.

1 horse-power	= 746 watts	
1 horse-power	= 7.604×10^6 gm. cm. per second	7.46×10^9 ergs per second
1 metric horse-power	= 7.5×10^6 gm. cm. per second	7.36×10^9 ergs per second
1 kilowatt	= 10^{10} ergs per second	
1 watt	= 10^7 ergs per second	

Mechanical Equivalent of Heat.

1 gram-degree C.	$= 4.281 \times 10^4$ gm. cm.	4.2×10^7 ergs
1 pound-degree F. (B.T.U.)	$= 1.942 \times 10^7$ gm. cm.	1.905×10^{10} ergs
1 pound-degree F.	= 777.4 foot-pounds.	

Miscellaneous.

1 U. S. gal.	= 231 cu. in. = 4 qts. = 8 pints
1 U. S. gal. (water)	= 8.313 lb. Av. = 58,418.1444 grains
1 bushel	= 2150.42 cu. in.
1 lb. Av. (water)	= 0.12029 gal.
1000 grains (water)	= 2.200 flu. oz.
1 pint	= 16 flu. oz.
1 flu. oz.	= 8 drams = 480 minims
1 flu. oz. (water)	= 456.392 grains = 1.0391 oz. Av.
1 oz. Av. (water)	= 0.9623 flu. oz.
1 lb. Av. (water)	= 0.12029 gal.
1 cu. in. (water)	= 252.892 grains
1 liter (water)	= 1 Kilo.
4 c.c. = 1 dram	= 1 teaspoonful
8 c.c. = 2 dram	= 1 dessert-spoonful
16 c.c. = 4 dram	= 1 table-spoonful

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